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I-B-G-Q-C: The Ward-Fixed Cone Product Globalized under a Declared Wick-Rotation Boundary Value, with Its Representative, and the Support Bridge for the Same Kernel

Version 9 (book edition t427u; file I-B-G-Q-C-v2): the contact Ward identity for three currents, and a four-current product off the total diagonal. Earlier: wavefront positivity and a three-current product with the Ward identity off the total diagonal

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Abstract

This is version 9 of I-B-G-Q-C (book edition t427u)-v2 (changes in Sections , , , , and); it answers request t253u. It answers Plan 4 about the vacuum-polarization sheets I-B-G-Q and I-B-G-Q-B, and it revises no other sheet.

I-B-G-Q-B cor:Q-globalobs shows that the Ward-fixed Lorentzian cone product of I-B-G-Q-B thm:Q-lorentzalpha does not globalize by analytic continuation, because a boundary-value prescription is missing. Here such a prescription is declared as *new data* (Declaration). The cone Green function G_α of I-B-G-Q-B def:Q-Galpha is continued to Lorentzian pairs as the boundary value from $\operatorname{Im} w > 0$, which is the limit along the Wick rotation. I-B-VP-2 lem:boundaryvalue supplies the boundary values.

The following is proved for $0 < \alpha \le 1$:

1. As a function of the invariant w, G_α is holomorphic off $(-\infty, -1]$ and finitely many real poles. The boundary value G_α^{bv} is a global distribution on off-axis pairs and solves the wave equation off the diagonal (Theorem).

2. Its singular part is explicit. On each image cone it is *exactly* the flat Feynman image term $1/(4\pi^2(\sigma_k + \mathrm{i}0))$: the residue in w is 1 for every α at generic θ (v3 correction: 2 at $\theta = 0$, where two images coincide; Proposition). On the diffracted front it is $(w + 1 + \mathrm{i}0)^{-1/2}$ times an analytic coefficient. This is an inverse-square-root singularity, not

a pole in *w*, except at shadow angles, where a pole $\frac{m}{2}(w+1)^{-1}$ is added (v3 correction, Proposition (b)); the source's description is corrected (Section). The prescription selects one of the two conjugate branches beyond the front.

3. The off-diagonal product built from G_α^{bv} glues with the regular-ball product into a globally conserved product. Its Ward seagull, which is the computed representative of the lift, is $\mathrm{i}(\alpha^{-2}-1)/(24\pi^2 r^2)\,\eta_{\mu\nu}\delta(x-x')$. (v3 correction) The conjugate prescription forces $-R_\alpha$, not R_α (Proposition). It is unique up to the conserved diagonal-supported tensors of I-B-G-Q-B thm:Q-actualdefect (Theorem).

4. Tested against I-B-G-Q-B thm:Q-notinherited, the representative is separated from the inherited codimension-two datum. For the global product, the singular-support test is inconclusive under (T0) alone; it separates for the benchmark datum (Proposition).

5. *Support bridge: yes.* For the same-prescription kernel, $\operatorname{supp}\operatorname{Im} G_\alpha^{\mathrm{bv}}$ lies in the closed completion relation $J=\{\sigma\le 0\}$ (v3 wording, Remark). So both support orderings give $fg\operatorname{Im} G_\alpha^{\mathrm{bv}}=0$, with no sphere–ball geometry needed (Theorem). Still, no time-ordered product is constructed, and the IX-E trace certificate is not verified. *New in version 2.* Plan t233u asked for a negative theorem: that the declared prescription yields no state and no time-ordered product for $\alpha\notin\{1/N\}$. That is *false*, and the correct statement is proved. G_α is the Euclidean kernel of the Friedrichs Laplacian (Lemma). For every $0<\alpha\le 1$, G_α^{bv} is the time-ordered two-point function of the static ground state ω_α of the Friedrichs dynamics, which is positive and has unitary time evolution (Theorem). The opposite prescription gives the anti-time-ordered function. Off the diagonal, the global product is the ground-state time-ordered bilinear (Corollary). The separation from the inherited codimension-two datum is extended to every (T0) datum whose singular support contains no hypersurface piece (Proposition). Composite time-ordered products, inherited identification, the IX-E-3 certificate and RG stay open. *New in version 3* (Section).

1. *Two-current time-ordered product (partial).* For the complex free field in ω_α, $\mathrm{T}(j_a j_b')$ is constructed as a form on a dense domain of smooth finite-particle vectors (Theorem). It satisfies causal factorization off the diagonal, and

an *operator* Ward identity, with total seagull $2i\eta_{ab}\delta(x-x')$(mathopen : $\varphi^*\varphi$mathclose : $+h$), $h=(\alpha^{-2}-1)/(48\pi^2r^2)$. Its c-number part is v2's seagull, and its operator coefficient is forced.

2. *Corrections of v2*: the residue is 2 at $\theta=0$; shadow angles produce a pole at $w=-1$ (Proposition); the conjugate prescription forces $-R_\alpha$ (Proposition); and J is the closed completion relation (Remark). The distribution-level existence results all stand. *New in version 4* (Section).

1. *Exact shadow set* (Proposition). Shadow angles are $\theta\equiv\pi \bmod 2\pi\alpha$. They exist iff $1/\alpha$ is not an odd integer; they are then $\pm\theta_s$, and the multiplicity is 2 exactly for $\alpha=1/N$, N even, $\theta=0$. So $\alpha=1/N$ with N odd, including the flat case, has no shadows.

2. *Ward-compatible diagonal terms* (Theorem). A diagonal term $\sum_\beta d^\beta_{ab}(x')\partial^\beta\delta(x-x')$, with operator-valued coefficients, preserves the Ward identity iff its symbol has the Koszul form $\xi^c\Lambda_{cab}$, Λ antisymmetric in c,a. Every order-zero term is excluded, for every tensor structure; at order two, $Q(\partial_a\partial_b-\eta_{ab}\mathrm{Box})\delta$ is allowed for every Q, but satisfies the Ward identity in both variables only for constant Q. This settles the v3 open row on operator-valued diagonal terms, relative to the Ward identity.

3. Audit of v3: no mathematical error; one imprecision (V3-N1: ``shadows occur for $\alpha=1/N$'' holds only for even N). *New in version 5* (Section).

1. *Two-sided classification* (Theorem). A polynomial 2-tensor symbol transverse in both indices is $\xi^c\xi^dR_{cadb}$ with R antisymmetric in each pair (degree ≥ 2), or $\xi^c\Theta_{cab}$ with Θ a constant 3-form (degree 1). This classifies the diagonal terms with point-independent coefficients that preserve both Ward identities; the Lorentz-covariant c-number ones are $(\partial_a\partial_b-\eta_{ab}\mathrm{Box})S(\mathrm{Box})\delta$, with $\lfloor p/2\rfloor$ of them under the order bound p. For point-dependent coefficients the leading symbol is classified.

2. *Symmetry* (Proposition). The two-current product is symmetric, and then satisfies both Ward identities, iff the local product of I-B-G-Q-B is symmetric; that is not established here.

3. *Higher products* stay open; the missing input is wavefront information on G^{bv}_α (Remark). Audit of v4: no error.

New in version 6 (Section).

1. *Symmetry* (Theorem). The local product T_{α} of I-B-G-Q-B is symmetric, by a direct computation from its defining formula. Hence the global product and the two-current product $\mathrm{T}(j_{a}j_{b}^{'})$ are symmetric and satisfy the Ward identity in both arguments.

2. *Point-dependent coefficients* (Theorem). The diagonal terms compatible with both identities are exactly $\xi^{c}(\xi^{d}-\partial_{x'}^{d})R_{cadb}+\xi^{c}\Theta_{cab}$ with Θ a divergence-free 3-form; the proof uses a second Koszul complex for $\xi-\partial_{x'}$ (Lemma).

3. *Three or more currents* stay open (Remark). Audit of v5: no error; one scope note (V5-N1). New exact checks Q22, Q23. *New in version 7* (Section).

1. *Wavefront positivity* (Lemma): off the diagonal every covector of $\mathrm{WF}(G_{\alpha}^{\mathrm{bv}})$ is $\xi_{w}\,\mathrm{d}w+\xi_{\theta}\,\mathrm{d}\theta$ with $\xi_{w}>0$, by analyticity in the angle. Pairing with the time-scaling field gives a product criterion for every graph of contractions (Lemma).

2. *Three-current product; partial* (Theorem): defined on all of $(M^{\circ})^{3}$, symmetric, time ordered and causally factorized; Furry cancellation of the c-number part; Ward identity off the total diagonal. The contact Ward identity and $n\geq 4$ stay open (Remark).

3. Audit of v6: no error; one scope note (V6-N1). New checks Q24, Q25 (exact) and Q26 (numerical corroboration). *New in version 8* (Section).

1. *Contact Ward identity* (Theorem): $\partial_{x_{1}}^{a}\,\mathrm{T}_{3}=0$ on all of $(M^{\circ})^{3}$. No diagonal term is needed and no new coefficient enters. The diagonal freedom of T_{3} is not classified.

2. *Four currents; partial* (Proposition): defined, with the Ward identity, off the total diagonal. The only obstruction is the C-even cycles (squares, seagull loops, seagull bubbles) at the total diagonal.

3. *Non-inheritance* for every (T0) datum whose singular support has no hypersurface piece (Proposition ; documentary).

4. Audit of v7: no error in proofs or statements; two scope errors in Remark (V7-N1, V7-E1). New checks Q27 (exact, combinatorial) and Q28 (exact). Higher time-ordered products, the full Epstein–Glaser axioms, inherited identification, the IX-E-3 certificate and RG stay open.

The plan's Oai282 wording agrees with I-B-G-Q-B thm:Q-282; the plan misattributes three v6 results to that theorem (Remark). The certificate stays RG-open. No quantum coefficient is admitted. Host C is not created; Host A and Host B stay separate. O.1 stays NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. The last reading bars (quantized observable, no external clock) stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal or Page curve is claimed. Gate 0 stays NOT_SUPPLIED, and thm:J is not substituted. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only; no proof transfer. Authorship requires confirmation. Nothing is pushed or registered.

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I-B-G-Q-C-v2: The Ward-Fixed Cone Product Globalized under a Declared Wick-Rotation Boundary Value, with Its Representative, and the Support Bridge for the Same Kernel Version 9 (book edition t427u; file I-B-G-Q-C-v2): the contact Ward identity for three currents, and a four-current product off the total diagonal. Earlier: wavefront positivity and a three-current product with the Ward identity off the total diagonal Ngo Cheung (authorship requires confirmation)

Author note. The registry author field is empty. The author line follows the title-page convention of the series, and the PDF author metadata is left empty.

Book edition note (t427u)

Standing Enter refusals, scope vocabulary (proved here/supplied input/conditional/ not constructed/not claimed), the four-layer instrument box, and Scholar bibliography policy are frozen in the companion chapter *I-B-G Book Ledger* v1. O.1 remains NOT_CLOSED; XVI-N is not discharged; XV-B is not opened; no Host C, Page/island, or spacetime-emergence claim; no quantum coefficient is admitted; wired=false; SOURCE_DATE_EPOCH=1792627200. This sheet is a companion manuscript record and is *not* claimed to be Google Scholar-indexed. The published Enter Calculus article is cited only for finite-instrument / operational interface discussions, never as a PDE premise.

Book expansion: separation table, contact Ward pointer, missing box

toprule Landmark Relation Separation

midrule I-B-G-Q-B Lorentzian cone product as source record Boundary value is declared new data

I-B-G-Q Gates / non-implications No gate closed by contact Ward alone

CERT Slot injectivity / minimax Certificate sheet; not a premise that admits χ_F

I-B-VP-2 Boundary-value lemma Supplied comparison for Wick rotation

bottomrule

Contact Ward and partial products

The contact Ward identity for three currents and the four-current product off the total diagonal are proved under the declared boundary-value hypothesis of this sheet. They are *partial* products: they do not supply the full interacting characteristic product, do not pass Q0–Q7 for the inherited target, and do not create Host C.

Still missing. Full characteristic product; interacting all-orders Ward; RG-closed certificate; Host C; quantum coefficient; Page/emergence. Finite+conserved≠non-running. Standing ledger verbatim (O.1, XVI-N, XV-B, wired=false).

Scope and standing ledger

Sources. I-B-G, I-B-VP-2, I-B-G-Q, I-B-G-Q-B, IX-E, IX-E-3, VII-H, Oai282-A and Oai282-B were fetched read-only from the site (HTTP 200) and copied to inputs/ with their sha256 values. All are source records, not premises. Oai282-A and Oai282-B (authors: OpenAI) are used only as comparisons, as in I-B-G-Q-B thm:Q-282. Every fact used in a proof is re-proved here, or is a standard fact about boundary values of holomorphic functions (I-B-VP-2 lem:boundaryvalue), or is quoted with its label and used only as a definition (the closed form I-B-G-Q-B def:Q-Galpha and the local product I-B-G-Q-B thm:Q-lorentzalpha).

Status ledger. toprule Item Status in this sheet

midrule Plan 4: boundary-value prescription **declared** as new data (Declaration)

Plan 4: seagull continued off the diagonal; representative **computed**: Theorems ,

Plan 4: test against Q-notinherited representative separated; global product: inconclusive in general, separated for the benchmark (Proposition)

Plan 4: IX support bridge, same prescription **yes** (Theorem), for the closed completion relation (Remark)

v3: two-current time-ordered product **constructed at form level, partial** (Theorem); higher products open

v3: multiplicity, shadow and conjugate corrections **made** (Propositions ,)

v4: shadow set **determined exactly** (Proposition)

v4: Ward-compatible diagonal terms (operator-valued, any order) **classified** relative to the Ward identity (Theorem); other normalization conditions not imposed

IX-E-3 trace certificate (TC) for the inherited operator not verified; unchanged

v5: diagonal terms compatible with both Ward identities **classified** for point-independent coefficients; leading symbol for all (Theorem); symmetry of the product reduced to the local product (Proposition)

v6: symmetry of T_{α}, T_{α}^{bv} and $\mathrm{T}(j_{a}j_{b}^{'})$; Ward identity in both arguments **proved** (Theorem), from the source's defining formulas

v6: diagonal terms with point-dependent coefficients, both identities **classified** (Theorem)

v7: three-current product **partial**: exists on all of $(M^{\circ})^{3}$, symmetric, causally factorized, c-number part zero (Furry), Ward identity off the total diagonal (Theorem); contact identity open

v8: contact Ward identity of T_{3} **proved** on all of $(M^{\circ})^{3}$, form level, no new coefficient (Theorem); diagonal freedom not classified

v8: four-current product **partial**: off the total diagonal with the Ward identity (Proposition); C-even cycles at Δ_{4} open

Time-ordered / state identification for $\alpha \notin \{1/N\}$ v1: not obtained. **v2: obtained at the two-point level**: the ground state of the Friedrichs dynamics (Theorem); v3: two-current product at form level (Theorem); higher composite products not constructed

Inherited characteristic coefficient not admitted

RG status of the certificate RG-open

VII-H free susceptibility not identified with any object here

O.1 NOT_CLOSED

XVI-N not discharged (N-Ins.4 unresolved); no Route-B successor

XV-B not opened

H2, I3, T3; J2 unresolved; Unresolved

U6-sel; Res02-A Q1–Q4, A1–A2 not moved; open

Host C not created; Host A and Host B separate

Last reading bars open (quantized observable, no external clock)

Einstein dynamics, derived metric, emergence, Page curve not claimed

Continuum acquisition negative certificate

bottomrule No quantum coefficient is admitted.

Setting. As in I-B-G-Q-B thm:Q-lorentzalpha, the Lorentzian cone is $-\mathrm{d}t^2+\mathrm{d}r^2+r^2\,\mathrm{d}\psi^2+\mathrm{d}z^2$ with $\psi\sim\psi+2\pi\alpha$, and $0<\alpha\le 1$ throughout. Let M° be the off-axis part. For $x,x'\in M^\circ$ let $\theta\in(-\pi\alpha,\pi\alpha]$ be the principal developed angle difference, and put

$$w=\frac{r^2+r'^{\wedge}2+\Delta z^2-\Delta t^2}{2rr'},\qquad \sigma=2rr'(w-\cos\theta),\qquad \sigma_k=2rr'(w-\cos(\theta+2\pi\alpha k)).$$

Here σ is the flat interval in the developed chart, and σ_k is the interval to the k-th image. Since $|\theta|\le\pi\alpha\le\pi$, the straight developed segment is a spatial geodesic, so $J=\{\sigma\le 0\}$ is the causal relation of the completed cone (v3 correction: for $\alpha=1$, $\theta=\pi$ the null pairs are joined only through the removed axis, so J is the closed completion relation; Remark). By I-B-G-Q-B def:Q-Galpha, $G_\alpha=g_\alpha(u,\theta)/(8\pi^2rr')$ with $w=\cosh u$, where

$$g_\alpha(u,\theta)=\frac{\sinh(u/\alpha)}{\alpha\sinh u\,(\cosh(u/\alpha)-\cos(\theta/\alpha))},$$

and $h_\alpha=g_\alpha-g_1$ gives the remainder H_α. The mode sum I-B-G-Q-B lem:Q-Galpha(a) uses only the modes $e^{-|n|u/\alpha}$, which are regular at the axis. This is the Friedrichs choice of I-B-G thm:green.

Changes to version 7

Version 7 (sha256 cf18a245ecb4e8d769aa755ef1c9b803fa1d64b4c4364516e20715cea61545f0) is retained read-only. Its results, labels and proofs are kept unchanged.

1. New Section : Theorem , Propositions and , Remark , and the verbatim standing ledger of plan t253u.

2. New checks Q27 (exact, combinatorial) and Q28 (exact).

3. Updated: the ledger, the open table, the plan corrections (PC21–PC24), the errors appendix (V7-N1, V7-E1), the drafts appendix and the release.

Changes to version 6

Version 6 (sha256 28899a7966522ab65e100c7ab41de2a4235e6409e821a13a6514ebfc966e23c9) is retained read-only. Its results, labels and proofs are kept unchanged.

1. New Section : Lemmas and , Theorem and Remark .

2. New checks Q24–Q26, with code/t37.py.

3. Updated: the ledger, the open table, the plan corrections (PC20), the errors appendix (V6-N1), the drafts appendix and the release.

Changes to version 5

Version 5 (sha256 b3c3286ec045fb348638d5f5e843561d66f17a21ed34c7ca5c189d3586953118) is retained read-only. Its results, labels and proofs are kept unchanged.

1. New Section : Theorem , Lemma , Theorem , Corollary and Remark .

2. New exact checks Q22 and Q23, with code/pt6.py.

3. Updated: the ledger, the open table, the plan corrections (PC19), the errors appendix (V5-N1), the drafts appendix and the release.

Changes to version 4

Version 4 (sha256 eac51c7dba521e222434ee929522d6a2f61ad001c6ee729deaee36239237dfc1) is retained read-only. Its results, labels and proofs are kept unchanged.

1. Section : Theorem , Proposition and Remark .

2. New checks Q20 and Q21, with code/dt5.py.

3. Updated: the ledger, the open table, the plan corrections (PC18), the errors appendix, the drafts appendix and the release.

Changes to version 3

Version 3 (sha256 d71665d01f0cc3f72454e994e4369a64e1c697f5936c83e113b329d5b94cec43) is retained read-only. Its results, labels and proofs are kept unchanged.

1. Section : Proposition , Theorem and Remark .

2. New checks Q17–Q19.

3. Updated: the ledger, the open table, the plan corrections (PC16–PC17), the errors appendix (V3-N1), the drafts appendix and the release. The standing-ledger sentence now reads verbatim ``Citations identify source records only; no proof transfer."

Changes to version 2

Version 2 (sha256 68d0c2c0ed48dff07cd650b375acde936720d3524360ad228d11142f9719b92d) is retained read-only. Its results are kept, with the corrections listed in (W2).

1. Section : Lemma and Theorem , the two-current time-ordered product (partial).

2. *Corrections*, made in place and marked ``(v3 correction)": Lemma (a),(b), Theorem (e), Theorem (d),(e), the setting paragraph and the abstract. They rest on Propositions and and Remark .

3. New checks Q13–Q16.

4. Updated: the ledger, the open table, the plan corrections (PC11–PC15), the errors appendix, the drafts appendix (E-D3) and the release.

Changes to version 1

Version 1 (sha256 96b71a0d067fd3a18a8de4e7694aea16b3abcaa0077819da8b8a510ae8aea4cf) is retained read-only, and all of its results are kept. New in version 2 (Section):

1. Definition , Lemma (Sobolev bound; $C_c^\infty \subset D(B^{-1/2})$) and Lemma ($G_\alpha$ and its continuation in ζ are the kernel of $(2B)^{-1}e^{-B\zeta}$).

2. Theorem : Declaration gives the time-ordered function of the static ground state, for every $0 < \alpha \leq 1$. This refutes the negative theorem requested by the plan (PC7).

3. Corollary identifies the global product off the diagonal. Proposition sharpens the separation test.

4. The ledger row, the open table, the plan corrections (PC7–PC10), the release and checks Q9–Q12 are updated. The non-claims of version 1 saying that a theorem ``does not imply a state, positivity, unitarity or a time-ordered identification" remain accurate for those theorems taken alone. Theorem supplies the identification using the additional Lemmas and .

The declared prescription and the global boundary value

Lemma (Analytic structure in *w*). *Fix* $\theta \in (-\pi\alpha, \pi\alpha)$ *and let* $\Gamma_\theta(w) = g_\alpha(\arccosh w, \theta)$, *with the principal branch of* arccosh *on* $\Omega = \mathbb{C} \setminus (-\infty, -1]$. *Let* $V_\theta = \{|\theta + 2\pi\alpha k| : k \in \mathbb{Z}\} \cap (0, \pi)$, *and add* 0 *to* V_θ *when* $\theta = 0$.

1. Γ_θ *is meromorphic on* Ω. *Its poles are the points* $w_\nu = \cos\nu$, $\nu \in V_\theta$. *They are real and simple, with residue exactly* 1 *(v3 correction: for* $\theta \neq 0$*; at* $\theta = 0$ *the poles with* $\nu \neq 0$ *have residue* 2, *Proposition (a)). Every image pole satisfies* $w_\nu \leq \cos\theta$.

2. *(v3 correction: away from shadow angles* $|\theta + 2\pi\alpha k| = \pi$, *where a pole* $\frac{m}{2}(w+1)^{-1}$ *is added, Proposition (b).)*

Near $w = -1$, $\Gamma_\theta(w) = A_\theta(w)(w+1)^{-1/2} + B_\theta(w)$, *with* A_θ, B_θ *holomorphic near* -1 *and* $A_\theta(-1) = \mathrm{i}\rho_\alpha(\theta)/\sqrt{2}$. *Here* $\rho_\alpha(\theta) = \operatorname{Res}_{u=\mathrm{i}\pi} g_\alpha = -\mathrm{i}\sin(\pi/\alpha)/(\alpha(\cos(\pi/\alpha) - \cos(\theta/\alpha)))$ *(I-B-G-Q-B thm:Q-global(b)). If* $\alpha = 1/N$ *then* $A_\theta \equiv 0$. *For* $\alpha \notin \{1/N\}$, $A_\theta(-1) \neq 0$ *for all but finitely many* θ.

3. Γ_θ *is real on* $(-1, \infty)$ *minus its poles.*

4. *On compact subsets of* $\{\operatorname{Re} w \in I\}$, *for a bounded interval* I, $|\partial_\theta^j \Gamma_\theta(w)| \le C_j (\operatorname{Im} w)^{-1-j}$ *for* $0 < \operatorname{Im} w \le 1$, *locally uniformly in* θ, *including as* $\theta \to \pm\pi\alpha$.

Proof. g_α is even in u, so Γ_θ is a function of $u^2 = \operatorname{arccosh}^2 w$, which is holomorphic on Ω. The principal branch maps Ω into $\{\operatorname{Re} u \ge 0,\ |\operatorname{Im} u| < \pi\}$, and the upper half-plane onto $\{\operatorname{Re} u > 0,\ 0 < \operatorname{Im} u < \pi\}$, since $\operatorname{Im}\cosh(a + \mathrm{i}b) = \sinh a \sin b$. Singularities can only come from zeros of $\sinh u$, which in this strip occur only at $u = 0$, and from zeros of $\cosh(u/\alpha) - \cos(\theta/\alpha)$, which are $u = \mathrm{i}(\pm\theta + 2\pi\alpha k)$. All of these are purely imaginary, so $w = \cos v$ is real. At $u = 0$ with $\theta \neq 0$ the quotient is regular.

(a) At $u_* = \mathrm{i}v$, $v \in V_\theta$, the residue of g_α in u is $1/\sinh u_*$ (I-B-G-Q-B thm:Q-global(b) for $k \neq 0$; for the direct pole $k = 0$ the residue equals that of $g_1 = 1/(\cosh u - \cos\theta)$, by I-B-G-Q-B thm:Q-lorentzalpha(a)). Since $\mathrm{d}w/\mathrm{d}u = \sinh u$, the residue in w is $\sinh u_*/\sinh u_* = 1$. If $\theta = 0$, then $w - 1 \sim u^2/2$ and $g_\alpha \sim 2/u^2$, so again the residue is 1. For $k \neq 0$, $|\theta + 2\pi\alpha k| \ge 2\pi\alpha - |\theta| \ge |\theta|$, so $\cos v \le \cos\theta$ whenever $v \in (0, \pi)$.

(b) g_α has a simple pole at $u = \mathrm{i}\pi$ with residue $\rho_\alpha(\theta)$, and its Laurent series in $\zeta = u - \mathrm{i}\pi$ has a regular part. Since $w = -\cosh\zeta = -1 - \zeta^2/2 - \ldots$, one has $\zeta = -\mathrm{i}\sqrt{2(w+1)}\,(1 + O(w+1))$ on the principal branch near $w = -1$, $w > -1$. So odd powers of ζ produce $(w+1)^{1/2}$ times holomorphic functions, and even powers produce holomorphic functions. The leading term is $\rho/\zeta = \mathrm{i}\rho/\sqrt{2(w+1)}$. For $\alpha = 1/N$, g_α is $2\pi\mathrm{i}$-periodic in u and even, hence even about $u = \mathrm{i}\pi$; it is then a function of ζ^2, so $A_\theta \equiv 0$. For $\alpha \notin \{1/N\}$, $\sin(\pi/\alpha) \neq 0$.

(c) For $w > 1$, u is real and every factor is real. For $-1 < w < 1$, $u = \mathrm{i}v$, and $\sinh(\mathrm{i}v/\alpha)/\sinh(\mathrm{i}v) = \sin(v/\alpha)/\sin v$ is real.

(d) Γ_θ is holomorphic in the upper half-plane. Near the real axis, its singularities are finitely many simple real

poles with residue 1, which can merge in pairs as $\theta \to \pm\pi\alpha$, together with the $(w+1)^{-1/2}$ term. Each contributes at most $C(\operatorname{Im} w)^{-1}$, and θ-derivatives move the poles smoothly, which costs one power per derivative. G_α is $2\pi\alpha$ -periodic and smooth in θ, so the bound is uniform up to $|\theta| = \pi\alpha$.

par *Does not imply.* The lemma does not imply anything about $\operatorname{Re} w < -1$ beyond the existence of the two one-sided limits. It uses the closed form of I-B-G-Q-B def:Q-Galpha as a definition and does not prove that G_α is the Green function of any inherited operator. par

Declaration (Boundary-value prescription; new data). On off-axis Lorentzian pairs, G_α is taken to be the distributional boundary value

$$G_\alpha^{\mathrm{bv}}(x, x') = \frac{1}{8\pi^2 rr'} \Gamma_\theta(w + \mathrm{i}0),$$

constructed by I-B-VP-2 lem:boundaryvalue with $z = w$ and smooth parameters. This is new data. The inputs, including I-B-G thm:green, I-B-G thm:closed and I-B-G thm:transfer, do not supply it; the need for it is I-B-G-Q-B cor:Q-globalobs(b).

par *Does not imply.* The declaration does not imply that G_α^{bv} is the two-point or time-ordered function of a state, or that it is positive, for $\alpha \notin \{1/N\}$. It is a prescription, not a selection. par

Theorem (Global boundary value and its singular structure). *Let* $0 < \alpha \le 1$.

1. *G_α^{bv} is a well-defined distribution on $M^\circ \times M^\circ$ minus the diagonal. It is symmetric, and its wavefront set lies in the half $\{\lambda \ge 0\}$ dual to w.*

2. *(Wick rotation.) Let G_ε be G_α evaluated at $\Delta t^2 \mapsto e^{-2\mathrm{i}\varepsilon}\Delta t^2$. Then $G_\varepsilon \to G_\alpha^{\mathrm{bv}}$ in $\mathscr{D}'$ as $\varepsilon \operatorname{downarrow} 0$. At $\varepsilon = \pi/2$ this is the Euclidean G_α.*

3. *On every Lorentzian regular ball, $G_\alpha^{\mathrm{bv}} = \Delta_F + H_\alpha^L$, with $\Delta_F = 1/(4\pi^2(\sigma + \mathrm{i}0))$ and H_α^L as in I-B-G-Q-B thm:Q-lorentzalpha.*

4. $\operatorname{Box}_x G_\alpha^{\mathrm{bv}} = \operatorname{Box}_{x'} G_\alpha^{\mathrm{bv}} = 0$ *off the diagonal.*

5. *(Singular part; v3 correction: with multiplicities m_k and the shadow term, Proposition (c).) Near each image cone $S_k = \{\sigma_k = 0\}$ with $|\theta + 2\pi\alpha k| \in (0, \pi)$, $G_\alpha^{\mathrm{bv}} - 1/(4\pi^2(\sigma_k + \mathrm{i}0))$ is smooth; the same holds at the direct cone with σ. Near the diffracted front $\{w = -1\}$,*

$$G_\alpha^{\mathrm{bv}} = \frac{A_\theta(w)\,(w + 1 + \mathrm{i}0)^{-1/2} + B_\theta(w)}{8\pi^2 r r'},$$

which is locally integrable. G_α^{bv} is real on $\{\sigma > 0\}$ and on $\{w > -1\}$ away from the cones. On $\{w < -1\}$, $\operatorname{Im} G_\alpha^{\mathrm{bv}} = -|w + 1|^{-1/2} \operatorname{Re} A_\theta/(8\pi^2 r r') + \dots$, which is nonzero for $\alpha \notin \{1/N\}$ and zero for $\alpha = 1/N$. The prescription selects one of the two complex-conjugate branches of I-B-G-Q-B thm:Q-global(c).

Proof. (a) Off the diagonal, the map $(x, x') \mapsto w$ has $\mathrm{d}w \neq 0$ at every point where Γ_θ is singular. Indeed $\mathrm{d}w = 0$ forces $\Delta t = \Delta z = 0$ and $r = r'$, hence $w = 1$, and $w = 1$ is a singular value only for $\theta = 0$, that is, on the diagonal. Near such a point choose coordinates (w, y), with y containing θ, r, r', z, z' and $t + t'$. Lemma (d) gives the hypotheses of I-B-VP-2 lem:boundaryvalue, which yields the distribution and the wavefront orientation I-B-VP-2 eq:BVorientation. Elsewhere Γ_θ is real-analytic. Symmetry holds because w is symmetric and g_α is even in θ.

(b) In the coordinates (w, y), $\Delta t^2 = r^2 + r'^2 + \Delta z^2 - 2rr'w$. The rotated invariant is $w_\varepsilon = w + (1 - e^{-2\mathrm{i}\varepsilon})\Delta t^2/(2rr')$, and $\operatorname{Im} w_\varepsilon = \sin(2\varepsilon)\Delta t^2/(2rr') \geq 0$. Moreover $\partial w_\varepsilon/\partial w = e^{-2\mathrm{i}\varepsilon}$. Write $\Gamma_\theta = \partial_w^m P$ with P a holomorphic primitive that is continuous up to the real axis (proof of I-B-VP-2 lem:boundaryvalue). Then $\Gamma_\theta(w_\varepsilon) = e^{2\mathrm{i}m\varepsilon}\partial_w^m[P(w_\varepsilon)]$. Since $P(w_\varepsilon) \to P(w + \mathrm{i}0)$ uniformly on compacts, this converges in $\mathscr{D}'$ to $\partial_w^m P(w + \mathrm{i}0) = \Gamma_\theta(w + \mathrm{i}0)$. At points with $\Delta t = 0$ there is no singularity off the diagonal, because then $w \geq 1$.

(c) On a regular ball, $G_\alpha = G_1 + H_\alpha$ with $G_1 = g_1/(8\pi^2 r r') = 1/(4\pi^2\sigma)$, and H_α is holomorphic near the relevant real w (I-B-G-Q-B thm:Q-lorentzalpha(a)). So the boundary value of H_α is H_α^L, and that of $1/\sigma$ is $1/(\sigma + \mathrm{i}0)$ because $rr' > 0$.

(d) For spacelike pairs, G_α is real-analytic and is the continuation of the Euclidean Green function, which is harmonic in each variable off the diagonal (I-B-G-Q-B lem:Q-Galpha(a)). So $\operatorname{Box} G = 0$ there. In the coordinates

(w,y), $\mathrm{Box}_x\,[\Gamma_\theta(w)/(8\pi^2 rr')] = (\mathscr{D}\Gamma)(w,y)$, where $\mathscr{D}$ is a second-order operator whose coefficients are rational in (w,y) and holomorphic near the real w-axis. $\mathscr{D}\Gamma$ is holomorphic in the upper half-plane and vanishes on a real interval of regularity, so it vanishes identically. Boundary values commute with ∂_w, ∂_y and multiplication by such coefficients. Hence $\mathrm{Box}_x\, G_\alpha^{\mathrm{bv}} = (\mathscr{D}\Gamma)(w + \mathrm{i}0) = 0$.

(e) By Lemma (a), $\Gamma_\theta - 1/(w - w_\nu)$ is holomorphic near w_ν, and $8\pi^2 rr'(w - w_\nu) = 4\pi^2\sigma_k$. The front term is Lemma (b). For $w < -1$, $(w + 1 + \mathrm{i}0)^{-1/2} = -\mathrm{i}|w + 1|^{-1/2}$. Reality on $\{\sigma > 0\}$ follows from Lemma (a),(c): there $w > \cos\theta \geq w_\nu$ and $w > -1$. The branch statement holds because the two one-sided limits are complex conjugate (I-B-G-Q-B thm:Q-global(c)). Checks Q2–Q4 and Q6 corroborate (a), (d) and (e) numerically.

par *Does not imply*. The theorem does not imply a state, positivity, unitarity or a time-ordered identification for $\alpha \notin \{1/N\}$, and it does not construct the Carslaw diffracted term of I-B-ECHO as an independent object. It does not identify G_α^{bv} with any inherited propagator. par

The global product and its representative

Write $\mathrm{B}[G]_{\mu\nu} = -(2\partial_\mu G\,\partial'_\nu G - 2G\,\partial_\mu\partial'_\nu G)$ for the off-diagonal Wick bilinear of I-B-G-Q-B def:Q-cone, with the Lorentzian normalization of I-B-G-Q-B thm:Q-lorentzalpha(e). Let T_α be the local Ward-fixed product of that theorem on regular balls, with seagull $s = 2\mathrm{i}H_\alpha^L(x,x)$.

Theorem (Globally conserved product and the computed representative). *Let* $0 < \alpha \leq 1$ *and* $\nu > 0$.

1. $\mathrm{B}[G_\alpha^{\mathrm{bv}}]$, *defined as the boundary value in* w *of* $\mathrm{B}[\Gamma_\theta/(8\pi^2 rr')]$, *is a distribution on* $M^\circ \times M^\circ$ *minus the diagonal. There is a unique* $T_\alpha^{\mathrm{bv}} \in \mathscr{D}'(M^\circ \times M^\circ)$ *that equals* T_α *on every product of regular balls and equals* $\mathrm{B}[G_\alpha^{\mathrm{bv}}]$ *off the diagonal.*

2. *(Global Ward identity.)* $\partial_x^\mu T^{\mathrm{bv}}_{\alpha,\mu\nu} = 0$ *on* $M^\circ \times M^\circ$.

3. *(Representative.) Among* $T_\alpha^{\mathrm{bv}} + a(x)\eta_{\mu\nu}\delta(x - x')$ *with* a *smooth, the Ward identity holds iff* $a = 0$. *So the diagonal term of the lift is forced to be*

$$\text{boxed } R_\alpha = s(x)\,\eta_{\mu\nu}\delta(x-x'), \qquad s(x) = \mathrm{i}\,\frac{\alpha^{-2}-1}{24\pi^2 r^2}$$

at every off-axis point. The Ward-preserving diagonal-supported modifications are exactly the conserved diagonal-supported tensors of I-B-G-Q-B thm:Q-actualdefect. Under (LC) and order $\le p$ these leave $\lfloor p/2 \rfloor$ slots (I-B-G-Q-B thm:Q-conecount).

4. *(Prescription dependence.) The opposite prescription $w - \mathrm{i}0$ yields a product T^-_α (v3 correction: with forced diagonal term $\overline{R}_\alpha = -R_\alpha$, not the same R_α; Proposition), and $T^{\mathrm{bv}}_\alpha - T^-_\alpha$ is supported in $J = \{\sigma \le 0\}$ off the diagonal.*

5. *(Seagull continued off the diagonal; v3 correction: image terms counted with multiplicity, shadow term added, Proposition (c).) On $U_\alpha = \{|\theta| < \pi\alpha\}$ the function $2\mathrm{i}H_\alpha$ continues to the distribution $2\mathrm{i}H^{\mathrm{bv}}_\alpha = 2\mathrm{i}(G^{\mathrm{bv}}_\alpha - \Delta_F)$. Its singular part is $\sum_k \frac{2\mathrm{i}}{4\pi^2(\sigma_k + \mathrm{i}0)}$ over the image cones with $|\theta + 2\pi\alpha k| < \pi$, plus $\frac{2\mathrm{i}A_\theta(w)}{8\pi^2 rr'}(w+1+\mathrm{i}0)^{-1/2}$ at the diffracted front. Its diagonal value is s.*

6. *(Running.) $v\partial_v T^{\mathrm{bv}}_\alpha = v\partial_v T^F_v$ is diagonal-supported and independent of α and of the prescription. The certificate stays RG-open.*

Proof. (a) $\mathrm{B}[\Gamma_\theta/(8\pi^2 rr')]$ is holomorphic in w in the upper half-plane, since products and derivatives of holomorphic functions are holomorphic, and ∂_x acts through the coordinates (w, y) with coefficients holomorphic in w. It obeys power bounds of the type in Lemma (d), so I-B-VP-2 lem:boundaryvalue applies, as in Theorem (a). The product of boundary values from the *same* half-plane is the boundary value of the product. (This is why the obstruction of C-D2's opposite-side example $(x+\mathrm{i}0)^{-1}(x-\mathrm{i}0)^{-1}$ does not arise.) On $V \times V$ minus the diagonal, $G^{\mathrm{bv}}_\alpha = \Delta_F + H^L_\alpha$ (Theorem (c)), so $\mathrm{B}[G^{\mathrm{bv}}_\alpha]$ equals the off-diagonal part of T_α, which is $\mathrm{B}[\Delta_F + H^L_\alpha]$ with the Feynman products of T^F_v. The two definitions therefore agree on overlaps. Distributions glue uniquely along an open cover (partition of unity), and the cover consists of $(M^\circ \times M^\circ) \setminus \mathrm{diag}$ and the products of regular balls around diagonal points.

(b) Off the diagonal, $\partial^\mu \mathrm{B}[G]_{\mu\nu} = -2(\mathrm{Box}\, G\,\partial'_\nu G - G\,\mathrm{Box}\,\partial'_\nu G) = 0$ by Theorem (d), applied first to the holomorphic

continuation and then to the boundary value. Near the diagonal this is I-B-G-Q-B thm:Q-lorentzalpha(e) with $s = 2iH^{L}_{\alpha}(x, x)$. Distributions vanishing on an open cover vanish.

(c) $\partial^{\mu}[a(x)\eta_{\mu\nu}\delta(x - x')] = a(x')\partial_{\nu}\delta(x - x')$, which vanishes iff $a = 0$. The value of s is 2i times I-B-G-Q-B thm:Q-lorentzalpha(d). The remaining statements are I-B-G-Q-B thm:Q-actualdefect(a)–(c) applied on regular balls, since T^{bv}_{α} satisfies (OD) there.

(d) The two boundary values agree wherever Γ_{θ} is real-analytic. By Lemma (a),(c) this includes all of $\{\sigma > 0\}$. (v3 correction: the diagonal terms are *not* identical; see Proposition .)

(e) Use Theorem (e). On U_{α} the principal θ is continuous, so Δ_{F} is a global term. Subtracting it removes the direct-cone singularity.

(f) Off the diagonal $\mathrm{B}[G^{\mathrm{bv}}_{\alpha}]$ does not depend on v. Near the diagonal use I-B-G-Q-B thm:Q-lorentzalpha(e) and I-B-G-Q-B thm:Q-run.

par *Does not imply.* The theorem does not imply a time-ordered product, a state or unitarity for $\alpha \notin \{1/N\}$, and it does not imply that T^{bv}_{α} is the inherited product or the inherited characteristic coefficient. The coefficients C^{γ} of the actual inherited lift are not computed. It does not fix the scale, does not identify anything with the free susceptibility of VII-H thm:release, and admits no coefficient. par

Proposition (Test against non-admission). *Let u° satisfy (T0) of I-B-G-Q-B thm:Q-notinherited on $\mathbb{R}^{4} \setminus W$, with W of dimension two.*

1. *For fixed x', no extension of u° equals $\Phi^{*}R_{\alpha}(\cdot, x')$ for a diffeomorphism Φ. The representative is therefore separated from the inherited datum, as in I-B-G-Q-B thm:Q-notinherited.*

2. *For the slice $T^{\mathrm{bv}}_{\alpha}(\cdot, x')$, the singular support contains the direct light cone of x', a three-dimensional set. So the singular-support argument of I-B-G-Q-B thm:Q-notinherited does not decide identification under (T0) alone.*

3. *If u° is smooth on $\mathbb{R}^{4} \setminus W$, as is the benchmark $|x_{\perp}|^{-2} \otimes 1$ of I-B-G-Q prop:v5-codim2(iii), then no extension of u°*

equals $\Phi^{*}T_{\alpha}^{\mathrm{bv}}(\cdot, x')$.

Proof. (a) sing supp $R_{\alpha}(\cdot, x') = \{x'\}$, a single point, while every extension of u° is singular on an open piece of W; proceed as in the cited proof. (b) Δ_F is singular on $\{\sigma = 0\}$, and $G_{\alpha}^{\mathrm{bv}} - \Delta_F$ is smooth near the direct cone of x' inside a regular ball. A three-dimensional set can contain a two-dimensional piece of $\Phi^{-1}(W)$, so the inclusion argument fails. (c) Φ^{-1} of a three-dimensional singular set is not contained in the two-dimensional W, so $\Phi^{*}T_{\alpha}^{\mathrm{bv}}$ is singular at some point of $\mathbb{R}^4 \setminus W$, where u° is smooth.

par *Does not imply.* The proposition is a typing statement. It does not assert that the inherited coefficient does not exist, it does not use the inherited coefficient, and it does not exclude identification under other typings. par

The support bridge for the same prescription

Lemma (Sphere–ball intersection). *For* $t > 0$, $s \geq 0$ *and* $R = |\mathbf{x} - \mathbf{y}|$ *in* $\mathbb{R}^3$, $\{|\mathbf{z} - \mathbf{x}| = t\} \cap \{|\mathbf{z} - \mathbf{y}| \leq s\} \neq \varnothing$ *iff* $|R - t| \leq s$, *that is, iff* $\max\{0, t - s\} \leq R \leq t + s$. *The condition* $|t - s| \leq R \leq t + s$ *is sufficient but not necessary. It fails for* $t = 1$, $s = 3$, $R = 0$, *where the sphere lies inside the ball.*

Proof. The distance from $\mathbf{y}$ to the sphere is $|R - t|$, and the set of distances from $\mathbf{y}$ to points of the sphere is the interval $[|R - t|, R + t]$. In the example, $|t - s| = 2 > 0 = R$, although the ball contains the whole sphere. Check Q7 corroborates this.

par *Does not imply.* This is IX-E-3 eq:sphere-ball, re-proved. It does not imply any support property of a kernel. par

Theorem (Support bridge: yes, for the same-prescription kernel). *Let* $0 < \alpha \leq 1$, *and let* $F, G \subset M^{\circ}$ *be compact with* $F \cap J^{-}(G) = \varnothing$ *and* $G \cap J^{-}(F) = \varnothing$, *where* $J^{\pm}$ *is the relation* $\{\sigma \leq 0\}$ *(v3 wording: the closed completion relation, Remark). Then*

$$\operatorname{supp} \operatorname{Im} G_{\alpha}^{\mathrm{bv}} \subseteq J \qquad \text{and} \qquad f(x)g(y)\operatorname{Im} G_{\alpha}^{\mathrm{bv}}(x, y) = 0$$

for all $f \in C_c^{\infty}$ *with support in F and* $g \in C_c^{\infty}$ *with support in G. There is a positive margin:* $\min_{F \times G} \sigma > 0$. *The same*

holds for $\operatorname{Im} \mathrm{B}[G_\alpha^{\mathrm{bv}}]$, *and for* $T_\alpha^{\mathrm{bv}} - T_\alpha^{-}$.

Proof. On the open set $\{\sigma > 0\}$, Γ_θ is real-analytic and real (Lemma (a),(c)), so its boundary value is that real function, and $\operatorname{Im} G_\alpha^{\mathrm{bv}} = 0$ there. Both orderings say that no pair in $F \times G$ is causally related, so $F \times G \subset \{\sigma > 0\}$, and by compactness σ has a positive minimum on it. A distribution that vanishes on a neighbourhood of $\operatorname{supp}(f \otimes g)$ is annihilated by multiplication with $f \otimes g$. The same argument applies to B, which is real-analytic and real wherever Γ_θ is, and to the difference of Theorem (d). Check Q4 corroborates this numerically.

par *Does not imply*. The theorem does not construct a time-ordered product T_2, a state, or a commutator algebra. It does not verify the boundary-trace certificate (TC) of IX-E-3 for the inherited operator $L = \mathrm{E}_0 C$, and it does not decide the IX-E obstruction alternative of IX-E thm:obstruction, whose hypotheses concern that operator. No quantum coefficient is admitted. par

Remark (Comparison with the flat IX-E kernel). The IX-E-2 kernel copied in IX-E-3 eq:imagK is the zero-Cauchy kernel on the half-space with initial surface Σ. Its imaginary part is nonzero on the reflected slab $|\tau| < R < t + s$, where pairs are spacelike, so for that kernel $\operatorname{supp} \operatorname{Im} K \subseteq J$. For this reason IX-E-3 thm:geometric and IX-E-3 thm:minimal need the sphere–ball geometry and the initial null spheres in the functional support of B_f. For the same-prescription kernel of Declaration there is no initial surface and no reflected part, and the bridge follows from the orderings alone (Theorem). The two kernels are different objects. Neither result transfers to the other, and the inherited (TC) remains as stated in IX-E-3. IX-E thm:algebraic is not affected.

Remark (The Oai282 wording). The plan's note says that ``running does not place the product outside that finite Ward certificate'', to be checked against I-B-G-Q-B thm:Q-282. The note *agrees* with that theorem. I-B-G-Q-B thm:Q-282(a),(b) shows that, in formal transcription of Oai282-A ass:ward, the certificate holds for every v with trivial contact assignment, and that the running is a contact term. Its last sentence refutes the opposite statement, ``the running places the product outside Oai282-A thm:ward and Family 282''. The v5 summary of I-B-G-Q-B says the same.

There is a discrepancy elsewhere: the plan attributes to I-B-G-Q-B thm:Q-282 three statements that it does not

contain. These are (i) the seagull $s = 2iH$ is global on the diagonal, (ii) the diffracted residue vanishes iff $\alpha = 1/N$, and (iii) every agreeing lift has defect class zero. They are I-B-G-Q-B cor:Q-globalobs(a), I-B-G-Q-B thm:Q-global(b) and I-B-G-Q-B thm:Q-actualdefect(a) (v6). This is recorded as PC2. The running of T^{bv}_{α} equals that of T^{F}_{v} (Theorem (f)), so the comparison is unchanged for the global product: still formal, with Oai282-B thm:classification used only as a comparison, and no Family-282 hypothesis supplied.

The prescription is the time ordering of the static ground state (version 2)

Plan t233u asks for a *negative* theorem: that Declaration yields no state and no time-ordered product for $\alpha \notin \{1/N\}$. That statement is false. The Friedrichs dynamics on the cone is static, and the declared boundary value is the time-ordered two-point function of its ground state, for every $0 < \alpha \le 1$. What remains open is recorded at the end of this section. Notation: $\Sigma^{\circ} = \{(r, \psi, z) : r > 0\}$ is the spatial section with the axis removed, carrying the volume $r\,\mathrm{d}r\,\mathrm{d}\psi\,\mathrm{d}z$, and $M^{\circ} = \mathbb{R}_t \times \Sigma^{\circ}$.

Definition (Friedrichs operator). A is the Friedrichs extension in $L^2(\Sigma^{\circ})$ of $-\Delta_{\Sigma}$ on $C^{\infty}_{c}(\Sigma^{\circ})$. Its form domain $Q(A)$ is the closure of $C^{\infty}_{c}(\Sigma^{\circ})$ in the norm $(\|u\|^2 + \|\nabla u\|^2)^{1/2}$. Put $B = A^{1/2}$.

par *Does not imply*. This is the standard construction. It does not identify A with any inherited operator of IX-E or I-B-G, and the other self-adjoint extensions are not discussed. par

Lemma (Sobolev bound and the domain of $B^{-1/2}$).

1. *There is C_{α} with $\|u\|_{L^6(\Sigma^{\circ})} \le C_{\alpha}\|A^{1/2}u\|$ for all $u \in Q(A)$. In particular* $\ker A = 0$.

2. *Every $f \in L^2 \cap L^{6/5}(\Sigma^{\circ})$ lies in $D(A^{-1/2})$ and in $D(B^{-1/2})$, with $\|A^{-1/2}f\| \le C_{\alpha}\|f\|_{6/5}$ and $\|B^{-1/2}f\|^2 \le \|f\|^2 + C_{\alpha}^2\|f\|_{6/5}^2$.*

Proof. (a) The map $\Phi(r, \psi, z) = (r\cos(\psi/\alpha), r\sin(\psi/\alpha), z)$ is a bijection from Σ° onto $\mathbb{R}^3$ minus a line. It distorts lengths by factors in $[1, \alpha^{-1}]$ and multiplies volume by α^{-1}. For $u \in C^{\infty}_{c}(\Sigma^{\circ})$, $u \circ \Phi^{-1}$ is Lipschitz with compact support in $\mathbb{R}^3$, so the Sobolev inequality of $\mathbb{R}^3$ transfers with constants depending only on α. The inequality passes to the form closure by Fatou's lemma along an approximating sequence. If $Au = 0$, then $\|u\|_6 = 0$. (b) For $\lambda > 0$ put

$u = (A + \lambda)^{-1} f \in Q(A)$. Then pair$f(A + \lambda)^{-1} f \le \|f\|_{6/5} \|u\|_6 \le C_\alpha \|f\|_{6/5} \|A^{1/2}(A + \lambda)^{-1} f\|$, and $\|A^{1/2}(A + \lambda)^{-1} f\|^2 \le$ pair$f(A + \lambda)^{-1} f$. Hence $\int (\mu + \lambda)^{-1} \, \mathrm{d}m_f(\mu) \le C_\alpha^2 \|f\|_{6/5}^2$ uniformly in λ, where m_f is the spectral measure of f. By monotone convergence, $\int \mu^{-1} \, \mathrm{d}m_f \le C_\alpha^2 \|f\|_{6/5}^2$, which is the first claim. The second follows from $\mu^{-1/2} \le 1 + \mu^{-1}$.

par *Does not imply*. A standard functional-analytic fact for the flat-metric cone. It concerns only the Friedrichs extension. par

Lemma (G_α is the Euclidean Friedrichs kernel). *For* $\operatorname{Re}\zeta > 0$*, the operator* $F(\zeta) = (2B)^{-1} e^{-B\zeta}$*, restricted to* $C_c^\infty(\Sigma^\circ)$*, has as integral kernel the function*

$$K_\zeta(y, y') = \frac{1}{8\pi^2 r r'} \Gamma_\theta\left(\frac{r^2 + r'^2 + \Delta z^2 + \zeta^2}{2 r r'}\right),$$

that is, the closed form of I-B-G-Q-B def:Q-Galpha with $\Delta\tau^2$ *replaced by* ζ^2*. The argument of* Γ_θ *never meets* $(-\infty, -1]$ *or the real poles. At real* $\zeta = \tau > 0$ *this is the Euclidean* G_α*.*

Proof. The angular Fourier series and the Fourier transform in z decompose A into the radial operators $\ell_\nu = -\partial_r^2 - r^{-1}\partial_r + \nu^2 r^{-2} + p^2$ with $\nu = |n|/\alpha$. The form closure respects this decomposition, so A is the direct sum of the Friedrichs extensions of the ℓ_ν. The Friedrichs extension of the Bessel operator is diagonalized by the Hankel transform with the regular solution $J_\nu(kr)$, for every $\nu \ge 0$ (Bruneau, Dereziński and Georgescu, *Ann. Henri Poincaré* 12 (2011)). Integrating $e^{-\zeta\omega}/(2\omega)$ with $\omega = \sqrt{k^2 + p^2}$ over p gives $K_0(k\zeta)$. Hence

$$K_\zeta = \frac{1}{2\pi\alpha} \sum_{n \in \mathbb{Z}} e^{\mathrm{i}n\theta/\alpha} \frac{1}{2\pi} \int_0^\infty k J_\nu(kr) J_\nu(kr') K_0(k\zeta) \, \mathrm{d}k \qquad (\Delta z = 0),$$

and similarly for $\Delta z \neq 0$. For real $\zeta = \tau > 0$, combining the τ- and z-integrals into the Fourier transform in the flat transverse plane and using $\int_0^\infty k J_\nu(kr) J_\nu(kr')(k^2 + q^2)^{-1} \, \mathrm{d}k = I_\nu(qr_<) K_\nu(qr_>)$, the n-th term becomes $(4\pi^2\alpha)^{-1} \int_0^\infty q J_0(q\rho) I_\nu(qr_<) K_\nu(qr_>) \, \mathrm{d}q$, with $\rho^2 = \tau^2 + \Delta z^2$. The classical representation $I_\nu(aq) K_\nu(bq) = \frac{1}{2}\int_{u_0}^\infty J_0(q\sqrt{2ab\cosh t - a^2 - b^2}) e^{-\nu t} \, \mathrm{d}t$, with $\cosh u_0 = (a^2 + b^2)/(2ab)$, together with Hankel inversion, gives $\int_0^\infty q J_0(q\rho) I_\nu(aq) K_\nu(bq) \, \mathrm{d}q = e^{-\nu u}/(2ab \sinh u)$ with $\cosh u = (a^2 + b^2 + \rho^2)/(2ab)$ (check Q9). Summing gives

the mode series of I-B-G-Q-B lem:Q-Galpha(a), which is G_α. For complex ζ, both sides are holomorphic in ζ on $\{\mathrm{Re}\,\zeta > 0\}$ when paired with test functions. The operator side is clear. On the closed-form side, $w = (r^2 + r'^2 + \Delta z^2 + \zeta^2)/(2rr')$ is real only for real ζ^2, which for $\mathrm{Re}\,\zeta > 0$ means $\zeta^2 > 0$ and $w > 1$. The two sides agree for $\zeta > 0$, so they agree everywhere by the identity theorem. Check Q12 compares the spectral integral with the closed form at three non-real values of ζ.

par *Does not imply*. The proof uses cited classical facts about Bessel operators and Bessel integrals, with numerical corroboration (Q9, Q12). It identifies a kernel; it does not identify any inherited object. par

Theorem (The declared prescription is the ground-state time ordering). *Let* $0 < \alpha \le 1$. *For* $f, g \in C_c^\infty(M^\circ)$ *write* $f_t = f(t, \cdot)$. *With* $\mathrm{pair}\cdot\cdot$ *the* L^2 *inner product, antilinear in the first slot, put*

$$W_\alpha(f,g) = \frac{1}{2}\mathrm{i}\mathrm{int}\,\langle B^{-1/2}\bar{f}_t, e^{-\mathrm{i}B(t-t')}B^{-1/2}g_{t'}\rangle\,\mathrm{d}t\,\mathrm{d}t',$$

$$W_\alpha^T(f,g) = \frac{1}{2}\mathrm{i}\mathrm{int}\,\langle B^{-1/2}\bar{f}_t, e^{-\mathrm{i}B|t-t'|}B^{-1/2}g_{t'}\rangle\,\mathrm{d}t\,\mathrm{d}t'.$$

Both are bilinear.

1. W_α *and* W_α^T *are distributions on* $M^\circ \times M^\circ$. W_α *is the two-point function of a quasifree state* ω_α *of the free field with dynamics* $\partial_t^2 + A$: $W_\alpha(\bar{f}, f) \ge 0$, *and* $W_\alpha(f,g) - W_\alpha(g,f) = -\mathrm{i}\mathrm{i}\mathrm{int}\,\langle \bar{f}_t, B^{-1}\sin(B(t-t'))g_{t'}\rangle\,\mathrm{d}t\,\mathrm{d}t'$, *which is* $-\mathrm{i}$ *times the causal propagator of* $\partial_t^2 + A$. *Time evolution is implemented by the unitary group* $e^{-\mathrm{i}Bt}$, *and* ω_α *is invariant under translations in t and z and rotations in* ψ.

2. W_α^T *is the time-ordered function of* ω_α: *it equals* $W_\alpha(x, x')$ *on* $\{t > t'\}$ *and* $W_\alpha(x', x)$ *on* $\{t < t'\}$.

3. *On* $(M^\circ \times M^\circ) \setminus \mathrm{diag}$, $G_\alpha^{\mathrm{bv}} = W_\alpha^T$. *Moreover* $G_\varepsilon(\varphi) \to W_\alpha^T(\varphi)$ *as* $\varepsilon \mathrm{downarrow} 0$ *for every* φ *in the algebraic tensor product* $C_c^\infty(M^\circ) \otimes C_c^\infty(M^\circ)$, *including test functions meeting the diagonal. So* W_α^T *is an extension of* G_α^{bv} *across the diagonal.*

4. *The opposite prescription* $w - \mathrm{i}0$ *gives* $\overline{G_\alpha^{\mathrm{bv}}}$, *the anti-time-ordered function of* ω_α. *In particular, for* $\alpha \notin \{1/N\}$ *the declared prescription yields a positive state with unitary dynamics, and the time-ordered two-point function of that*

state.

Proof. (a) By Lemma (b), $t \mapsto B^{-1/2} f_t$ is continuous and compactly supported in L^2, with norm controlled by $\sup_t (\|f_t\|_2 + \|f_t\|_{6/5})$. So both forms are separately continuous on $C_c^\infty(M^\circ)$, and the kernel theorem makes them distributions. Positivity holds because $W_\alpha(\bar{f}, f) = \frac{1}{2}\|\int e^{iBt} B^{-1/2} f_t \, dt\|^2$, using that A commutes with complex conjugation. The commutator is the antisymmetric part of e^{-iBs}. The construction of a quasifree state from a positive two-point function whose antisymmetric part is i times a real symplectic form is standard.

(b) On $\{t > t'\}$, $|t - t'| = t - t'$. On $\{t < t'\}$, $\langle B^{-1/2}\bar{f}_t, e^{-iB(t'-t)} B^{-1/2} g_{t'}\rangle$ is the swapped Wightman pairing, because A is real.

(c) Put $\zeta_\varepsilon(s) = e^{i(\pi/2 - \varepsilon)}|s|$. Then $\zeta_\varepsilon(s)^2 = -e^{-2i\varepsilon} s^2$, so G_ε of Theorem (b) is $K_{\zeta_\varepsilon(t-t')}(y, y')$ at $t \neq t'$, by Lemma , with $\operatorname{Re}\zeta_\varepsilon = |s|\sin\varepsilon > 0$. For fixed $\varepsilon > 0$, G_ε is locally integrable on $M^\circ \times M^\circ$: near the diagonal $||\Delta y|^2 + \zeta_\varepsilon^2| \geq c_\varepsilon(|\Delta y|^2 + \Delta t^2)$, and ρ^{-2} is integrable in four dimensions. By Fubini and Lemma , $G_\varepsilon(f \otimes g) = \mathrm{i}\int\int \langle \bar{f}_t, F(\zeta_\varepsilon(t - t'))g_{t'}\rangle \, dt \, dt'$. Now $|\langle \bar{f}_t, F(\zeta) g_{t'}\rangle| \leq \frac{1}{2}\|B^{-1/2} f_t\| \|B^{-1/2} g_{t'}\|$ for $\operatorname{Re}\zeta \geq 0$, and $e^{-B\zeta_\varepsilon(s)} \to e^{-iB|s|}$ strongly by the spectral theorem. Dominated convergence gives $G_\varepsilon(f \otimes g) \to W_\alpha^T(f \otimes g)$. If $\operatorname{supp} f \times \operatorname{supp} g$ avoids the diagonal, the left side also tends to $G_\alpha^{\mathrm{bv}}(f \otimes g)$ by Theorem (b). Every $\varphi \in C_c^\infty((M^\circ \times M^\circ) \setminus \mathrm{diag})$ is a limit in $\mathscr{D}$ of finite sums of such products: cover the support by finitely many product sets $U_i \times V_i$ off the diagonal, use a partition of unity, and approximate in each $C_c^\infty(U_i \times V_i)$ by elements of $C_c^\infty(U_i) \otimes C_c^\infty(V_i)$. Two distributions agreeing on a dense set are equal.

(d) Γ_θ is real on $(1, \infty)$, so $\Gamma_\theta(\bar{w}) = \overline{\Gamma_\theta(w)}$ by the reflection principle (check Q10). The anti-time-ordered function is $\overline{W_\alpha^T}$ for real test functions.

par *Does not imply*. The theorem is a two-point statement for the free field with the Friedrichs dynamics. It does not construct Wick-ordered currents as operators, an interacting or composite time-ordered product, or the inherited state. It does not prove the Hadamard property on M° beyond the regular balls, where Theorem (c) gives

the flat Feynman singularity plus a smooth term. It does not select the Friedrichs extension or the ground state over other extensions or states (thermal states, for example). It does not verify the IX-E-3 certificate, and it admits no coefficient. The certificate stays RG-open. par

Corollary (The global product is a ground-state time-ordered bilinear off the diagonal). *Off the diagonal,* $T_\alpha^{\mathrm{bv}} = \mathrm{B}[W_\alpha^T]$. *On* $\{t > t'\}$ *this equals* $\mathrm{B}[W_\alpha]$, *the Wick-contraction expression for the current two-point function in* ω_α, *and on* $\{t < t'\}$ *it equals the swapped expression. The diagonal term is the forced representative* R_α *of Theorem (c). It is unchanged by the identification and unique up to the conserved diagonal-supported tensors of I-B-G-Q-B thm:Q-actualdefect.*

Proof. Theorem (a) and Theorem (b),(c). The products are boundary values from one side, so they are the products of the boundary values (proof of Theorem (a)).

par *Does not imply*. ``Wick-contraction expression" names the formula $\mathrm{B}[W]$; the Wick currents are not constructed as operators. The corollary does not imply that T_α^{bv} is the inherited product or carries the inherited characteristic coefficient, it does not fix the scale, and it leaves RG open. par

Proposition (Separation by singular-support dimension). *Let* u° *satisfy (T0) of I-B-G-Q-B thm:Q-notinherited on* $\mathbb{R}^4 \setminus W$, *with W closed of dimension two. Suppose* $\operatorname{sing\,supp} u^\circ$ *contains no nonempty relatively open subset of a* C^1 *hypersurface; for instance, it has Hausdorff dimension* < 3. *Then no extension of* u° *equals* $\Phi^* T_\alpha^{\mathrm{bv}}(\cdot, x')$ *for a diffeomorphism* Φ. *Conversely, if* u° *is singular on an open piece of a hypersurface, the singular-support test is silent.*

Proof. The slice singular support contains an open piece S of the direct light cone of x' minus its vertex, which is a smooth hypersurface (Proposition (b)). $\Phi^{-1}(S)$ is again a piece of a C^1 hypersurface. Its intersection with $\mathbb{R}^4 \setminus W$ is nonempty and relatively open, since a closed two-dimensional W contains no open piece of a three-dimensional manifold. That intersection would lie in $\operatorname{sing\,supp} u^\circ$, contrary to the hypothesis. For the converse, the inclusion argument has nothing to compare.

par *Does not imply*. This is a typing statement that sharpens Proposition (c). The benchmark has empty singular support off *W*. The proposition does not assert that the inherited coefficient does not exist, and it does not use it.

par

What remains open after Theorem.

• A time-ordered product of composite (Wick) currents as operator-valued distributions, and the Hadamard property on all of M°, which is not globally hyperbolic (I-B-G prop:nonGH).

• Identification of ω_{α} or A with an inherited state or operator.

• The IX-E-3 certificate (TC) for the inherited operator.

• The RG status.

Version 3: corrections of v2, and the two-current time-ordered product

Multiplicity and shadow angles

Version 2's Lemma (a),(b) and Theorem (e) are correct only for *generic* θ. Two exceptional configurations were missed.

Proposition (Coincident images and shadow angles). *Let* $0 < \alpha \leq 1$ *and* $|\theta| < \pi\alpha$.

1. *Multiplicity. For* $\theta \neq 0$ *the angles* $|\theta + 2\pi\alpha k|$, $k \in \mathbb{Z}$, *are pairwise distinct, and each pole of Lemma (a) has residue* 1. *For* $\theta = 0$ *the images* k *and* $-k$ *give the same angle* $2\pi\alpha|k|$. *Every pole* $w = \cos(2\pi\alpha k)$ *with* $k \geq 1$ *and* $2\pi\alpha k < \pi$ *therefore has residue* 2. *For example,* $\alpha = \frac{1}{3}$ *and* $\theta = 0$ *give* $\Gamma_0(w) = \frac{1}{w-1} + \frac{2}{w+1/2}$.

2. *Shadow angles. Suppose* $|\theta + 2\pi\alpha k| = \pi$ *for some* k. *Then the representation of Lemma (b) fails:* Γ_θ *has, in addition to the* $(w+1)^{-1/2}$ *term, a pole at* $w = -1$. *Precisely,* $\Gamma_\theta(w) = c_\theta(w+1)^{-1} + A_\theta(w)(w+1)^{-1/2} + B_\theta(w)$ *near* $w = -1$, *with* A_θ, B_θ *holomorphic and* $c_\theta = m/2 \neq 0$. *Here* $m \in \{1, 2\}$ *is the number of* k *with* $|\theta + 2\pi\alpha k| = \pi$. *The formula for* $\rho_\alpha(\theta)$ *in Lemma (b) has a vanishing denominator there. Shadows occur for* $\alpha = 1/N$, *for example* $\alpha = \frac{1}{2}$ *and* $\theta = 0$, *where* $\Gamma_0 = \frac{1}{w-1} + \frac{1}{w+1}$. *They occur also for other* α, *for example* $\alpha = 0.7$ *and* $\theta = -0.4\pi$. *The claim*

$A_\theta \equiv 0$ *for* $\alpha = 1/N$ *stands, but the remainder* B_θ *may then have a pole.*

3. *The distributional statements survive: Lemma (d), Theorem (a)–(d) and Theorem (a)–(c),(f) hold as stated. Theorem (e) and Theorem (e) hold after two changes. First, near a point of* S_k *subtract* $m_k/(4\pi^2(\sigma_k + \mathrm{i}0))$*, where* $m_k \in \{1, 2\}$ *is the number of images whose cones pass through the point. Second, at shadow configurations add the term* $c_\theta/(8\pi^2 rr'(w + 1 + \mathrm{i}0))$ *at the front. That term is not locally integrable but is a well-defined boundary value.*

Proof. (a) $|\theta + 2\pi\alpha k| = |\theta + 2\pi\alpha k'|$ with $k \neq k'$ forces $\theta = -\pi\alpha(k + k')$. With $|\theta| < \pi\alpha$ this forces $k' = -k$ and $\theta = 0$. At $\theta = 0$ the denominator $\cosh(u/\alpha) - 1$ has double zeros at $u = 2\pi\mathrm{i}\alpha k$, and $\sinh(u/\alpha)$ has simple zeros there, so g_α has a simple pole in u with residue $2/\sinh u$. That gives residue 2 in w. For $\alpha = 1/N$ and $\theta = 0$, $g = NU_{N-1}(w)/(T_N(w) - 1)$, with Chebyshev polynomials. Partial fractions give the examples (check Q13), and check Q14 confirms residue 2 numerically at $\alpha = 0.3$. (b) If $\theta + 2\pi\alpha k = \pm\pi$, then $u = \mathrm{i}\pi$ is a zero of $\cosh(u/\alpha) - \cos(\theta/\alpha)$ and also of $\sinh u$. So g_α has a double pole at $u = \mathrm{i}\pi$, Write $\zeta = u - \mathrm{i}\pi$, so that $\sinh u = -\zeta + O(\zeta^3)$ and $w + 1 = -\zeta^2/2 + O(\zeta^4)$. Let $D(u) = \cosh(u/\alpha) - \cos(\theta/\alpha)$.

• If $\sin(\pi/\alpha) \neq 0$ ($m = 1$), then D has a simple zero with $D'(\mathrm{i}\pi) = \mathrm{i}\sin(\pi/\alpha)/\alpha$, the numerator $\sinh(\mathrm{i}\pi/\alpha) = \mathrm{i}\sin(\pi/\alpha)$ is nonzero, and $g_\alpha = -\zeta^{-2} + O(\zeta^{-1})$.

• If $\sin(\pi/\alpha) = 0$ ($m = 2$), then D has a double zero and the numerator a simple zero, and $g_\alpha = -2\zeta^{-2} + O(\zeta^{-1})$. Since $\zeta^{-2} = -1/(2(w + 1)) + O(1)$, this gives $c_\theta = m/2$. The ζ^{-1} term gives the square-root term, as in v2's proof. For $\alpha = \frac{1}{2}$, $\theta = 0$: $g = 2\cosh u/\sinh^2 u = 2w/(w^2 - 1)$. For $\alpha = 0.7$, $\theta = -0.4\pi$: $\theta + 2\pi\alpha = \pi$, and check Q15 finds $\lim(w + 1)\Gamma_\theta = \frac{1}{2}$. (c) Lemma (d) only needs $|\Gamma_\theta| \leq C(\mathrm{Im}\, w)^{-1}$, which also holds for a double image pole (residue 2) and for the extra simple pole at $w = -1$. The remaining statements use (d), Theorem (c) and the regularity on $\{\sigma > 0\}$, none of which is affected.

par *Does not imply*. The proposition corrects singular-structure formulas. It does not imply a new state or product, and it does not change the forced representative R_α. par

The conjugate prescription

Proposition (Conjugate contact). *Let T^{-}_{α} be any distribution on $M^{\circ} \times M^{\circ}$ that equals $\mathrm{B}[\overline{G^{\mathrm{bv}}_{\alpha}}]$ off the diagonal, satisfies the Ward identity, and has a diagonal term of the form $a(x)\eta_{\mu\nu}\delta(x - x')$ relative to $\overline{T^{\mathrm{bv}}_{\alpha}}$. Then $T^{-}_{\alpha} = \overline{T^{\mathrm{bv}}_{\alpha}}$. Its forced diagonal term is $\overline{R_{\alpha}} = -R_{\alpha}$, that is, $\bar{s} = -s = -\mathrm{i}(\alpha^{-2} - 1)/(24\pi^2 r^2)$. Hence v2's Theorem (d) (``the same $R_{\alpha}$'') and the abstract's ``unchanged by the prescription'' are wrong.*

Proof. B has real coefficients, so $\overline{\mathrm{B}[G]} = \mathrm{B}[\overline{G}]$. The real operator ∂^{μ} commutes with conjugation, so $\overline{T^{\mathrm{bv}}_{\alpha}}$ satisfies the Ward identity. Uniqueness is Theorem (c), applied to the difference. Since s is purely imaginary, $\bar{s} = -s$.

par *Does not imply*. The proposition concerns the conjugate (anti-time-ordered) product only. Theorem still holds for $T^{\mathrm{bv}}_{\alpha} - T^{-}_{\alpha}$, because the diagonal difference $2R_{\alpha}$ is supported in the diagonal. No coefficient is admitted. par

Remark (The relation *J*; missing axis). v2 called $J = \{\sigma \le 0\}$ ``the causal relation of the cone''. For $\alpha < 1$ and $|\theta| < \pi\alpha < \pi$, the developed segment avoids the axis. For $\alpha = 1$ and $\theta = \pi$, however, a null pair with $\sigma = 0$ is joined only through the removed axis, and it is not causally related in M°. So J is the *closed completion relation*: the closure of the causal relation of M° in the completed cone. Theorem is correct with J defined as $\{\sigma \le 0\}$, which is how it is stated, and its hypothesis is to be read with that J. With the causal relation of M° in place of J, the hypothesis is weaker and the theorem is not claimed.

The two-current time-ordered product at form level (partial)

Version 2 constructed the two-point function and its time ordering, but not composite products. Here the two-current product of a free complex field is constructed as a sesquilinear form on a dense domain, with an operator-valued Ward identity. This answers part of the plan's ``composite Wick-current time-ordered product''. It is labelled *partial*: higher products, the Epstein–Glaser axioms beyond the properties listed, and all identifications stay open.

Setting. Let $\mathfrak{h} = L^2(\Sigma^{\circ})$ and let $\mathscr{F}$ be the Fock space of two independent copies of the quasifree field of Theorem .

Combine them into a complex field φ, with $\omega_\alpha(\varphi(x)\varphi^*(x')) = W_\alpha(x,x')$ and $\omega_\alpha(\varphi\varphi) = 0$, normalized as in Theorem . Let $\mathscr{D} \subset \mathscr{F}$ be the span of the finite-particle vectors whose one-particle factors lie in $E_{[\varepsilon,R]}(B)\mathfrak{h}$ for some $0 < \varepsilon < R < \infty$. Here E is the spectral projection of B. $\mathscr{D}$ is dense, because $\ker B = 0$ (Lemma).

Lemma (Smooth matrix elements). *For $\Psi, \Psi' \in \mathscr{D}$, the matrix elements of the normal-ordered products* mathopen : $\varphi^*(x)\varphi(x')$mathclose :, mathopen : $\varphi(x)\varphi^*(x')$mathclose :, *of their derivatives, and of* mathopen : $j_a(x)j_b(x')$mathclose :, *are smooth functions on $M^\circ \times M^\circ$. Each solves the field equation $(\partial_t^2 + A)u = 0$ in each variable.*

Proof. The matrix elements are finite sums of products of the functions $(t,y) \mapsto (e^{\mp iBt}B^{-1/2}u)(y)$ and their complex conjugates, with $u \in E_{[\varepsilon,R]}(B)\mathfrak{h}$. Such u lies in $D(A^k)$ for every k, and $t \mapsto e^{\mp iBt}B^{-1/2}u$ is smooth into every $D(A^k)$. On Σ° the operator A acts as $-\Delta$ on its domain, so interior elliptic regularity gives $D(A^\infty) \subset C^\infty(\Sigma^\circ)$, with locally uniform bounds. The field equation holds because $(\partial_t^2 + B^2)e^{\mp iBt} = 0$.

par *Does not imply.* The lemma concerns the off-axis region. It says nothing about behaviour at the axis, and it does not select the Friedrichs extension. par

Write $j_a = \mathrm{i}(\varphi^*\partial_a\varphi - \partial_a\varphi^*\varphi)$ (Wick ordered) and $G = W_\alpha^T$, a distribution on all of $M^\circ \times M^\circ$ (Theorem (c)). Put N = mathopen : $\varphi^*(x)\varphi(x')$mathclose : +mathopen : $\varphi(x)\varphi^*(x')$mathclose :, $n(x) = N(x,x)$ = 2mathopen : $\varphi^*\varphi$mathclose : (x), and write subscripts for derivatives, primed for the second variable. Let Box $= \eta^{ab}\partial_a\partial_b = -(\partial_t^2 + A)$ on smooth off-axis functions. The kernel $K(s) = e^{-\mathrm{i}B|s|}/(2B)$ of W_α^T satisfies $(\partial_s^2 + B^2)K = -\mathrm{i}\delta(s)$, so $\mathrm{Box}_x G = c\,\delta(x - x')$ with $c = \mathrm{i}$. With this c, the c-number part of (c) below reproduces v2's seagull $s = 2\mathrm{i}H_\alpha^L(x,x)$.

Theorem (Two-current time-ordered product; partial). *Define, as a form on $\mathscr{D} \times \mathscr{D}$ with values in $\mathscr{D}'(M^\circ \times M^\circ)$,*

$$\mathrm{T}(j_a j_b') = \text{mathopen : } j_a(x)j_b(x')\text{mathclose : } + S_{ab} + 2c\,\eta_{ab}\,\delta(x - x')\,\text{mathopen : } \varphi^*\varphi\text{mathclose : } (x) + T^{\mathrm{bv}}_{\alpha,ab}\,1,$$

$$S_{ab} = -[G_a N_{b'} + G_{b'} N_a - G_{ab'} N - G N_{ab'}].$$

1. *Existence. Every term is a well-defined distribution in each matrix element.* S_{ab} *is a distribution times a smooth function, by Lemma , and* T_{α}^{bv} *is v2's global product.*

2. *Time ordering off the diagonal. On* $\{t > t'\}$, $\mathrm{T}(j_a j_b')$ *equals the Wick expansion of the operator product* $j_a(x) j_b(x')$ *in* ω_α. *On* $\{t < t'\}$ *it equals that of* $j_b(x') j_a(x)$. *Off the diagonal it is symmetric under* $(a, x) \leftrightarrow (b, x')$.

3. *Operator Ward identity.* $\partial_x^a \mathrm{T}(j_a j_b') = 0$ *on all of* $M^\circ \times M^\circ$. *The total diagonal term is*

$$2c\,\eta_{ab}\,\delta(x - x')(\mathrm{mathopen} : \varphi^* \varphi \mathrm{mathclose} : (x) + h(x)), \qquad h = \frac{\alpha^{-2} - 1}{48\pi^2 r^2},$$

whose c-number part $2ch = s$ *is v2's seagull.*

4. *Forced operator seagull. Among the modifications* $\mathrm{T} + c_{ab}(x)\,\delta(x - x')\mathrm{mathopen} : \varphi^* \varphi \mathrm{mathclose} : (x)$, *with* c_{ab} *smooth, the Ward identity holds iff* $c_{ab} = 0$. *The c-number freedom is that of Theorem (c).*

Proof. (a) Lemma , Theorem (c), and Theorem (a).

(b) For quasifree states, Wick's theorem holds as an identity of forms on $\mathscr{D}$. The product $j_a(x) j_b(x')$ is the sum of the fully normal-ordered term, the one-contraction terms with the contraction $W_\alpha(x, x')$, and the two-contraction term $\mathrm{B}[W_\alpha]$. Expanding $j_a j_b' = -(\varphi^* \partial_a \varphi - \partial_a \varphi^* \varphi)(x)(\varphi^* \partial_b \varphi - \partial_b \varphi^* \varphi)(x')$, the one-contraction terms of type $\varphi(x)\varphi^*(x')$ are

$$-[G_a \mathrm{mathopen} : \varphi^*(x)\partial_b \varphi(x')\mathrm{mathclose} : - G_{ab'}\mathrm{mathopen} : \varphi^* \varphi'\mathrm{mathclose} : - G\,\mathrm{mathopen} : \partial_a \varphi^* \partial_b \varphi'\mathrm{mathclose} : + G_{b'}\mathrm{mathopen} : \partial_a \varphi^* \varphi'\mathrm{mathclose} :]$$

. Those of type $\varphi^*(x)\varphi(x')$ are the same with $\varphi \leftrightarrow \varphi^*$, since j changes sign under that exchange. Together they give S_{ab} with W_α in place of G. On $\{t > t'\}$, $G = W_\alpha(x, x')$ (Theorem (b)). The two-contraction term is $\mathrm{B}[W_\alpha] = T_\alpha^{\mathrm{bv}}$ there (Corollary). The case $t < t'$ is the same with the roles exchanged. The diagonal term vanishes off the diagonal. At $t = t'$ and $x \neq x'$, both orders agree, because the commutator function vanishes on equal-time pairs off the diagonal.

(c) $\partial^a \mathrm{mathopen} : j_a j_b' \mathrm{mathclose} : = 0$, because the Wick-ordered current is conserved by Lemma . $\partial^a T_{\alpha, ab}^{\mathrm{bv}} = 0$ by

Theorem (b). For S, a direct computation (check Q16, symbolic) gives

$$\partial^a S_{ab} = -[\mathrm{Box}\, G\, N_{b'} - (\mathrm{Box}\, G)_{b'} N + G_{b'} \mathrm{Box}\, N - G(\mathrm{Box}\, N)_{b'}] = -c[\delta N_{b'} - \partial_{b'} \delta N],$$

using $\mathrm{Box}\, N = 0$. Here $\delta = \delta(x - x')$, and products of δ and $\partial\delta$ with smooth N are understood distributionally. Now $\delta F(x, x') = \delta F(x', x')$, and $\partial_{b'} \delta F = \partial_{b'}(\delta F|_{\mathrm{diag}}) - \delta\, (\partial_{b'} F)|_{\mathrm{diag}}$, where $\partial_{b'}$ of a function of x' alone is the total derivative. Writing $d_b n = (N_b + N_{b'})|_{\mathrm{diag}}$, this gives

$$\delta N_{b'} - \partial_{b'} \delta N = \delta\, (N_{b'} - N_b)|_{\mathrm{diag}} - \partial_{b'} \delta\, n(x').$$

The first term vanishes: $(N_{b'} - N_b)|_{\mathrm{diag}} = -\mathrm{i} j_b + \mathrm{i} j_b = 0$, where the two contributions come from

mathopen : $\varphi^* \varphi'$ mathclose : and mathopen : $\varphi \varphi^{*'}$ mathclose :. Hence

$\partial^a S_{ab} = c\, n(x')\, \partial_{b'} \delta = -2c$ mathopen : $\varphi^* \varphi$ mathclose : $(x')\, \partial_b \delta(x - x')$. On the other hand,

$\partial^a_x [F(x) \delta(x - x')] = F(x') \partial^a_x \delta(x - x')$ for smooth F. So the seagull term has divergence

$2c$ mathopen : $\varphi^* \varphi$ mathclose : $(x') \partial_b \delta$, which cancels. The value of h is $s/(2c)$ with s from Theorem (c).

(d) $\partial^a_x [c_{ab}(x) m(x) \delta(x - x')] = c_{ab}(x') m(x') \partial^a \delta(x - x')$, with m = mathopen : $\varphi^* \varphi$ mathclose :. This vanishes as a form iff $c_{ab}\, m = 0$. The matrix elements of m do not vanish identically: take a one-particle $\Psi = \Psi'$ with a nonzero wavefunction.

par *Does not imply*. The theorem is a free-field, two-current, form-level statement on the off-axis region, and it is *partial*. It does not construct higher time-ordered products, does not verify the full Epstein–Glaser axioms (causal factorization is shown only for the two-current product off the diagonal), and does not classify operator-valued diagonal terms with derivatives of δ. It does not prove the Hadamard property on M°, does not identify ω_α, A or T with inherited objects, does not verify the IX-E-3 certificate (TC), and leaves RG open. No quantum coefficient is admitted. par

What remains open after v3. Higher and interacting time-ordered products, and the full Epstein–Glaser axioms; operator-valued diagonal ambiguities beyond $c_{ab} \delta$ mathopen : $\varphi^* \varphi$ mathclose :; the Hadamard property on all of M°;

identification with inherited data; the IX-E-3 certificate (TC); and RG.

Routing note (not registered). Nothing has been registered, pushed or written to the site.

Version 4: the exact shadow set, and the classification of Ward-compatible diagonal terms

Version 4 answers request t245u (``work harder''). It settles two items left by v3: the location of the shadow angles of Proposition (b), and the v3 open row ``operator-valued diagonal terms with derivatives of δ: not classified''. It also audits v3 (Appendix).

The shadow set

Proposition (Exact shadow set). *Let $0 < \alpha \le 1$ and $|\theta| < \pi\alpha$. Call θ a shadow angle if $|\theta + 2\pi\alpha k| = \pi$ for some $k \in \mathbb{Z}$.*

1. *θ is a shadow angle iff $\theta \equiv \pi \bmod 2\pi\alpha$. Shadow angles exist iff there is an integer k with $|\frac{1}{2\alpha} - k| < \frac{1}{2}$, that is, iff $1/\alpha$ is not an odd integer. The integer k is then unique, and the shadow angles are exactly $\pm\theta_s$, $\theta_s = \pi - 2\pi\alpha k$.*

2. *The multiplicity m of Proposition (b) is 2 iff $\alpha = 1/N$ with N even and $\theta = 0$. In every other shadow configuration $m = 1$.*

3. *In particular: for $\alpha = 1/N$ with N odd (including $\alpha = 1$) there is no shadow angle; for $\alpha = 1/N$ with N even the only shadow angle is $\theta = 0$; for $\alpha \notin \{1/N\}$ with $1/\alpha$ not an odd integer there are two, $\pm\theta_s \neq 0$, each with $m = 1$.*

Proof. (a) $\theta + 2\pi\alpha k = \pm\pi$ is $\theta \equiv \pm\pi \equiv \pi \bmod 2\pi\alpha$ after replacing k by $-k$ and θ by $-\theta$ where needed; precisely, the set of shadow angles is $\{\pi - 2\pi\alpha k\} \cup \{-\pi + 2\pi\alpha k\}$ intersected with $(-\pi\alpha, \pi\alpha)$, which is symmetric. Now $|\pi - 2\pi\alpha k| < \pi\alpha$ iff $|\frac{1}{2\alpha} - k| < \frac{1}{2}$. Such a k exists iff $\frac{1}{2\alpha} \notin \frac{1}{2} + \mathbb{Z}$, i.e. $1/\alpha$ is not an odd integer, and it is unique because the inequality is strict. (b) $m = 2$ means $\theta + 2\pi\alpha k = \pi$ and $\theta + 2\pi\alpha k' = -\pi$, so $\alpha(k - k') = 1$, i.e. $\alpha = 1/N$ with $N = k - k'$. By (a), shadows then require N even, $k = N/2$, and $\theta_s = \pi - \pi = 0$. Conversely, $\alpha = 1/N$, N even and $\theta = 0$ give $k = \pm N/2$. (c) restates (a) and (b).

par *Does not imply*. The proposition locates the configurations at which v3's correction applies. It changes no

distributional statement, it selects no prescription, and it does not change the forced representative R_{α}. Check Q17 compares it with a brute-force search and with the residue $\lim (w+1)\Gamma_{\theta} = \frac{1}{2}$ at a non-1/N shadow. par

Diagonal terms compatible with the operator Ward identity

Let $\mathcal{V}$ be the real vector space of sesquilinear forms on $\mathcal{D} \times \mathcal{D}$ whose matrix elements are smooth functions on M° (for example mathopen : $\varphi^{*}\varphi$mathclose :, mathopen : j_{a}mathclose :, and c-numbers). A *diagonal term* of order $\leq p$ is

$$D_{ab} = \sum_{|\beta| \leq p} d^{\beta}_{ab}(x') \, \partial^{\beta}_{x} \delta(x - x'), \qquad d^{\beta}_{ab} \in \mathcal{V} \otimes \mathbb{C}.$$

Every distribution supported on the diagonal with smooth coefficients has this form locally, by Schwartz's structure theorem. Put $P_{ab}(x'; \xi) = \sum_{\beta} d^{\beta}_{ab}(x') \xi^{\beta}$, a polynomial in $\xi \in \mathbb{R}^{4}$ with coefficients in $\mathcal{V} \otimes \mathbb{C}$; indices are raised with η.

Theorem (Ward-compatible diagonal terms).

1. $\mathrm{T}(j_{a} j_{b}') + D_{ab}$ *satisfies the Ward identity* $\partial^{a}_{x}(\cdot) = 0$ *of Theorem (c) iff, for every* x' *and* b*,*

$$P_{ab}(x'; \xi) = \xi^{c} \Lambda_{cab}(x'; \xi), \qquad \Lambda_{cab} = -\Lambda_{acb},$$

for polynomials Λ_{cab} *with coefficients in* $\mathcal{V} \otimes \mathbb{C}$ *(smooth in* x' *when* P *is).*

2. *In particular the order-*0 *part of* D *vanishes:* $d^{0}_{ab} = 0$*, for every tensor structure and every coefficient in* $\mathcal{V}$*. This extends Theorem (d) from* c_{ab}mathopen : $\varphi^{*}\varphi$mathclose : *to all order-zero terms.*

3. *The homogeneous Ward-compatible* $P_{\cdot b}$ *of degree* q *(for fixed* b *and fixed coefficient) form a space of dimension* $4\frac{q+3}{3} - \frac{q+4}{3}$*:* $0, 6, 20, 45, 84$ *for* $q = 0, \ldots, 4$*. At degree* 1 *they are* $P_{ab} = \Lambda_{cab}\xi^{c}$ *with constant antisymmetric* $\Lambda_{\cdot\cdot b}$*; at degree* 2 *they include* $Q(\xi_{a}\xi_{b} - \eta_{ab}\xi^{2})$*, that is* $D_{ab} = Q(x')(\partial_{a}\partial_{b} - \eta_{ab}\mathrm{Box})\delta(x - x')$*, for every* $Q \in \mathcal{V}$*, in particular* Q = mathopen : $\varphi^{*}\varphi$mathclose :.

4. *Two-sided identity. For* $D_{ab} = Q(x')(\partial_{a}\partial_{b} - \eta_{ab}\mathrm{Box})\delta(x - x')$ *one also has* $\partial^{b}_{x} D_{ab} = 0$ *iff* Q *is constant. Hence an*

operator-valued Q such as mathopen : $\varphi^{*}\varphi$mathclose : *is compatible with the Ward identity in x only; among these terms, those compatible with both identities are the c-number constants times* $(\partial_a\partial_b - \eta_{ab}\mathrm{Box})\delta$.

Proof. (a) Since $\partial_x^a[d(x')\partial_x^\beta\delta(x-x')] = d(x')\partial^a\partial_x^\beta\delta(x-x')$ and T already satisfies the identity, the condition is $\sum_\beta d_{ab}^\beta(x')\partial^a\partial^\beta\delta = 0$. The distributions $f(x')\partial_x^\gamma\delta(x-x')$ with distinct γ are linearly independent over smooth coefficients (pair with $\varphi(x)\psi(x')$, φ a polynomial near the support of ψ). So the condition is the polynomial identity $\xi^a P_{ab}(x';\xi) = 0$ for each x' and b. Fix x' and b, and let $R = \mathbb{C}[\xi_0,\dots,\xi_3]$. The Koszul complex of the regular sequence $\xi_0,\dots,\xi_3$ is exact, so the kernel of $R^4 \to R$, $(P_a)\mapsto\xi^aP_a$, is the image of Λ^2R^4, i.e. $P_a = \xi^c\Lambda_{ca}$ with Λ antisymmetric. Tensoring with the coefficient space $\mathscr{V}\otimes\mathbb{C}$, which is free over $\mathbb{C}$, preserves exactness. Smoothness in x': the Koszul map is a fixed linear map between finite-dimensional spaces of polynomials of bounded degree with coefficients, so a smooth right inverse on its image exists (choose one by linear algebra on the coefficient vectors). (b) $\xi^c\Lambda_{cab}$ has no constant term. (c) For $q \ge 0$ the map from vectors of degree-q polynomials to degree-$(q+1)$ polynomials, $(P_a)\mapsto\xi^aP_a$, is onto (every monomial of degree $q+1\ge1$ is divisible by some ξ^a). So the kernel has dimension $4\frac{q+3}{3} - \frac{q+4}{3}$. At $q=1$ the kernel consists of $\Lambda_{ca}\xi^c$ with Λ constant antisymmetric. At $q=2$, $\xi^a(\xi_a\xi_b - \eta_{ab}\xi^2) = \xi^2\xi_b - \xi_b\xi^2 = 0$. The numbers are check Q18. (d) $\partial_x^b[Q(x')\partial_x^\gamma\delta(x-x')] = (\partial^bQ)(x')\partial_x^\gamma\delta - Q(x')\partial_x^b\partial_x^\gamma\delta$. For $\partial^\gamma \to (\partial_a\partial_b - \eta_{ab}\mathrm{Box})$ the second term is $Q\,\partial^b(\partial_a\partial_b - \eta_{ab}\mathrm{Box})\delta = Q(\partial_a\mathrm{Box} - \partial_a\mathrm{Box})\delta = 0$. The first is $(\partial^bQ)(\partial_a\partial_b - \eta_{ab}\mathrm{Box})\delta$, which vanishes iff $\xi_a\xi^c\partial_cQ - \xi^2\partial_aQ = 0$ as a polynomial in ξ (linear independence again). For $d \ne a$ the coefficient of ξ_d^2 is $-\eta^{dd}\partial_aQ$, so $\partial_aQ = 0$ for every a (check Q19, symbolic). Conversely a constant Q gives 0.

par *Does not imply*. The theorem classifies diagonal terms of the given form relative to the Ward identity only. It does not impose or verify the other Epstein–Glaser normalization conditions (symmetry, scaling degree, unitarity), does not select a term, and constructs no higher time-ordered product. It is a free-field, form-level statement on the off-axis region. It identifies nothing with inherited data, leaves the IX-E-3 certificate unverified and RG open. No quantum coefficient is admitted. par

Remark (Effect on the representative). The forced c-number seagull R_α of Theorem is an order-zero diagonal term and is unaffected: by Theorem (b) no order-zero term can be added without breaking the Ward identity, in agreement with Theorem (c). The freedom that remains is of order ≥ 1 and of the Koszul form, and with two-sided Ward identities the operator-valued part of the order-two family is excluded. The scaling-degree restriction sd ≤ 4 of Epstein–Glaser would bound p; it is not imposed here.

Audit of version 3. Proposition , Proposition , Lemma and Theorem were rechecked; no mathematical error was found. One imprecision is recorded (V3-N1, Appendix).

Routing note (not registered). Nothing has been registered, pushed or written to the site.

Version 5: the two-sided Ward identity for the two-current product

Version 5 answers request t246u (``go even further''). Version 4 classified the diagonal terms compatible with the Ward identity in x (Theorem) and left the two-sided classification open. Here it is closed for coefficients that do not depend on the point (Theorem), and the leading symbol is classified for all coefficients. Symmetry of the product itself is reduced to a property of the local product of I-B-G-Q-B (Proposition). Higher time-ordered products, with three or more currents, are *not* constructed; the obstruction is recorded in Remark . Notation is that of Section : $D_{ab} = \sum_\beta d^\beta_{ab}(x')\,\partial_x^\beta \delta(x-x')$, with symbol $P_{ab}(x';\xi) = \sum_\beta d^\beta_{ab}(x')\xi^\beta$ and coefficients in $\mathscr{W} = \mathscr{V}\otimes\mathbb{C}$.

Theorem (Doubly transverse symbols; two-sided Ward classification). *Let* $R = \mathbb{C}[\xi_0,\dots,\xi_3]$.

1. *(Algebra.) Let* $P_{ab} \in R\otimes\mathscr{W}$ *be homogeneous of degree q. Then* $\xi^a P_{ab} = 0$ *for all b and* $\xi^b P_{ab} = 0$ *for all a if and only if*

$$q=0: P=0; \qquad q=1: P_{ab} = \xi^c \Theta_{cab}; \qquad q\geq 2: P_{ab} = \xi^c\xi^d R_{cadb},$$

where $\Theta \in \Lambda^3\otimes\mathscr{W}$ *is constant and totally antisymmetric, and* $R_{cadb}\in R\otimes\mathscr{W}$ *is homogeneous of degree* $q-2$ *with* $R_{cadb} = -R_{acdb} = -R_{cabd}$. *A general P is doubly transverse iff each homogeneous part is.*

2. *(Two-sided identity, constant coefficients.) If the coefficients* d^β_{ab} *do not depend on* x', *then* $\mathrm{T}(j_a j'_b) + D_{ab}$ *satisfies*

the Ward identity in x and $\partial^{b}_{x'} D_{ab} = 0$ if and only if P is as in (a).

3. *(Lorentz-covariant c-number terms.) If in addition the coefficients are c-numbers and $P_{ab}(\Lambda\xi) = \Lambda_{a}{}^{c}\Lambda_{b}{}^{d}P_{cd}(\xi)$ for all proper orthochronous Lorentz transformations, then $P_{ab} = (\xi_a\xi_b - \eta_{ab}\xi^2)S(\xi^2)$ for a polynomial S, that is, $D_{ab} = (\partial_a\partial_b - \eta_{ab}\mathrm{Box})S(\mathrm{Box})\delta(x - x')$, up to the sign convention of $\partial \leftrightarrow \xi$. Under the order bound $\deg P \le p$ (scaling degree $\le 4 + p$) these terms form a space of dimension $\lfloor p/2 \rfloor$ for $p \ge 2$ and 0 for $p < 2$.*

4. *(Symmetry.) With constant coefficients, D is symmetric under $(a, x) \leftrightarrow (b, x')$ iff $P_{ab}(\xi) = P_{ba}(-\xi)$. The degree-1 terms $\xi^{c}\Theta_{cab}$ and the terms of (c) are symmetric.*

5. *(Leading symbol, any coefficients.) If the coefficients depend smoothly on x' and D is compatible with both identities, then the top-degree part P^{M} of P is, for each x', as in (a), and $\xi^{b}P^{m-1}_{ab} = \partial^{b}_{x'}P^{m}_{ab}$ for every $m \le M$.*

Proof. (a) Write $K_{\cdot}$ for the Koszul complex of the regular sequence $\xi_0, \dots, \xi_3$ over R, with $K_p = \Lambda^{p}R^{4}$ and differential d (contraction with ξ). It is exact in degrees $p \ge 1$, and $\mathrm{im}\,(K_1 \to K_0)$ is the ideal (ξ). Tensoring over R with a free module, or over $\mathbb{C}$ with $\mathscr{W}$, preserves this. View $P \in K_1 \otimes_R K_1$ (first index a, second b). The two conditions say $(d \otimes 1)P = 0$ and $(1 \otimes d)P = 0$.

By exactness at $K_1 \otimes K_1$ in the first factor, $P = (d \otimes 1)\Lambda$ with $\Lambda \in K_2 \otimes K_1$, homogeneous of degree $q - 1$. Then $(d \otimes 1)(1 \otimes d)\Lambda = (1 \otimes d)P = 0$, so by exactness at $K_2 \otimes K_0$, $(1 \otimes d)\Lambda = (d \otimes 1)\Theta$ with $\Theta \in K_3 \otimes K_0$ of degree $q - 1$.

If $q \ge 2$, Θ has degree ≥ 1, so its entries lie in (ξ) and $\Theta = (1 \otimes d)\Theta'$ with $\Theta' \in K_3 \otimes K_1$. Put $\Lambda' = \Lambda - (d \otimes 1)\Theta'$. Then $(d \otimes 1)\Lambda' = P$ and $(1 \otimes d)\Lambda' = (d \otimes 1)\Theta - (d \otimes 1)(1 \otimes d)\Theta' = 0$. By exactness in the second factor, $\Lambda' = (1 \otimes d)R$ with $R \in K_2 \otimes K_2$. So $P = (d \otimes d)R$, which in indices is $P_{ab} = \xi^{c}\xi^{d}R_{cadb}$ with the stated antisymmetries.

If $q = 1$, Λ is constant, $P_{ab} = \xi^{c}M_{cab}$ with $M_{cab} = -M_{acb}$, and $\xi^{b}P_{ab} = \xi^{b}\xi^{c}M_{cab} = 0$ forces $M_{cab} = -M_{cba}$ as well (the symmetric part in (c, b) of a constant tensor contracted with $\xi^{b}\xi^{c}$ vanishes only if it is 0). A tensor antisymmetric in (c, a) and in (c, b) is totally antisymmetric. If $q = 0$, $\xi^{a}P_{ab} = 0$ with constant P forces $P = 0$. Conversely, each displayed form is doubly transverse by the antisymmetries. The conditions are homogeneous, which gives the last

sentence. Check Q20 confirms the dimension count for $q \le 4$ by exact rank computation.

(b) The x-identity is $\xi^a P_{ab} = 0$ (proof of Theorem (a)). For the x'-identity, $\partial_{x'}^b[d(x')\partial_x^\beta \delta(x - x')] = (\partial^b d)(x')\partial_x^\beta \delta - d(x')\partial_x^b \partial_x^\beta \delta$. With constant d the first term vanishes, and by linear independence of the $\partial_x^\gamma \delta(x - x')$ the condition is $\xi^b P_{ab} = 0$. Apply (a).

(c) By the first fundamental theorem of invariant theory for the proper orthochronous Lorentz group (documentary record of a standard result), a polynomial covariant 2-tensor of one vector is $A(\xi^2)\eta_{ab} + B(\xi^2)\xi_a \xi_b$; the possible term $\varepsilon_{abcd}\xi^c \xi^d$ vanishes. Then $\xi^a P_{ab} = (A + B\xi^2)\xi_b = 0$ gives $A = -\xi^2 B$, so $P = B(\xi^2)(\xi_a \xi_b - \eta_{ab}\xi^2)$. In the form of (a), $R_{cadb} = -B\,(\eta_{cd}\eta_{ab} - \eta_{cb}\eta_{ad})$. Under $\deg P \le p$, $\deg_\xi S(\xi^2) \le p - 2$, so S ranges over polynomials in ξ^2 of degree $\le \lfloor (p-2)/2 \rfloor$, a space of dimension $\lfloor p/2 \rfloor$ for $p \ge 2$. This agrees with the count of I-B-G-Q-B thm:Q-conecount, which is quoted and not used.

(d) With constant coefficients, the swap sends $d_{ab}^\beta \partial_x^\beta \delta(x - x')$ to $d_{ba}^\beta \partial_{x'}^\beta \delta(x' - x) = (-1)^{|\beta|} d_{ba}^\beta \partial_x^\beta \delta(x - x')$. For $\xi^c \Theta_{cab}$: $P_{ba}(-\xi) = -\xi^c \Theta_{cba} = \xi^c \Theta_{cab}$. The terms of (c) are even in ξ and symmetric in (a, b). Check Q21.

(e) The x'-identity reads $\xi^b P_{ab} = \partial_{x'}^b P_{ab}$ by the computation in (b). Compare homogeneous parts: the right side has degree $\le M$, the left side's top part is $\xi^b P_{ab}^M$ of degree $M + 1$, so $\xi^b P_{ab}^M = 0$; the lower parts give the recursion. Together with $\xi^a P_{ab}^M = 0$, (a) applies to $P^M(x'; \cdot)$ for each x'.

par *Does not imply*. The theorem classifies admissible diagonal terms; it does not select one, does not fix the scale, and does not identify any of them with the inherited product. Part (c) rests on a cited invariant-theory result as a documentary record, and Lorentz covariance of coefficients at a point of M° is a stated condition, not a property of the cone. Part (e) is a necessary condition only: the full classification for x'-dependent operator-valued coefficients is not given. No quantum coefficient is admitted. It does not imply a higher time-ordered product. par

Proposition (Symmetry of the two-current product).

1. *T$(j_a j_b')$ of Theorem is symmetric under $(a, x) \leftrightarrow (b, x')$ on all of $M^\circ \times M^\circ$ if and only if T_α^{bv} is.*

2. *T_α^{bv} is symmetric if and only if the local Ward-fixed product T_α of I-B-G-Q-B thm:Q-lorentzalpha(e) is symmetric on every product of regular balls.*

3. *If* $\mathrm{T}(j_a j_b')$ *is symmetric, it satisfies the Ward identity in* x' *as well:* $\partial_{x'}^b \mathrm{T}(j_a j_b') = 0$.

Proof. (a) mathopen : $j_a(x) j_b(x')$mathclose : is symmetric, since normal-ordered bosonic products commute. In S_{ab} the swap exchanges $G_a N_{b'}$ with $G_{b'} N_a$ and fixes $G_{ab'} N$ and $G N_{ab'}$, because $G = W_\alpha^T$ and N are symmetric in (x, x'). The operator seagull $2c\eta_{ab}\delta(x - x')$mathopen : $\varphi^* \varphi$mathclose : (x) is symmetric because $\delta(x - x')F(x) = \delta(x - x')F(x')$. So $\mathrm{T} - \mathrm{T}^{\mathrm{swap}} = T_\alpha^{\mathrm{bv}} - (T_\alpha^{\mathrm{bv}})^{\mathrm{swap}}$ times 1. (b) B[G] is symmetric off the diagonal, since G is. The swapped distribution $(T_\alpha^{\mathrm{bv}})^{\mathrm{swap}}$ equals B[G_α^{bv}] off the diagonal and T_α^{swap} on products of regular balls. If T_α is symmetric, it satisfies the characterization of Theorem (a), so it equals T_α^{bv} by uniqueness. The converse is restriction. (c) $\partial_{x'}^b \mathrm{T}_{ab}(x, x') = [\partial_y^b \mathrm{T}_{ba}(y, x)]_{y = x'} = 0$ by Theorem (c).

par *Does not imply.* Symmetry of T_α is a property of the source's local product; it is not proved here, so neither the symmetry nor the two-sided identity of T is established. The proposition does not construct a time-ordered product satisfying the Epstein–Glaser axioms and transfers no proof from I-B-G-Q-B. It does not imply a higher time-ordered product. par

Remark (Three or more currents). A form-level product of three currents needs, beyond terms that v3's Lemma controls, the fully contracted loop $G(x_1, x_2)G(x_2, x_3)G(x_3, x_1)$ and its derivatives. In v2 the two-current loop is defined because both factors are boundary values in the *same* variable w (proof of Theorem (a)). In the three-current loop the factors are boundary values in three different invariants w_{12}, w_{23}, w_{31}, and their product needs wavefront information on G_α^{bv} (Hadamard or Feynman type on all of M°), including at the diffracted front. That information is not supplied by any input. So higher products, and with them the full Epstein–Glaser axioms, stay open.

Audit of version 4. Theorem , Proposition and Remark were rechecked. No error was found. Theorem (d) is the special case $Q(\xi_a \xi_b - \eta_{ab}\xi^2)$ of Theorem (b),(e): with constant Q it is the $S = Q$ case of (c), and with non-constant

Q the recursion of (e) fails at $m = M$, in agreement with v4.

Routing note (not registered). Nothing has been registered, pushed or written to the site.

Version 6: symmetry, the two-sided identity, and point-dependent coefficients

Version 6 answers request t247u (``go further''). Version 5 reduced the symmetry of the two-current product to a property of the local product T_α of I-B-G-Q-B thm:Q-lorentzalpha(e) (Proposition), and classified the admissible diagonal terms only for constant coefficients and at the leading symbol (Theorem). Here the symmetry of T_α is proved from its defining formula, so $\mathrm{T}(j_a j_b')$ is symmetric and satisfies the Ward identity in both arguments (Theorem). The classification is completed for coefficients that depend smoothly on the point (Theorem). Products of three or more currents stay open (Remark).

Symmetry of the local product

Write sw for the exchange $(\mu, x) \leftrightarrow (\nu, x')$ on tensor distributions on $V \times V$: $(\mathrm{sw}\, T)_{\mu\nu}(x, x') = T_{\nu\mu}(x', x)$, the pull-back by the diffeomorphism $(x, x') \mapsto (x', x)$ combined with the index exchange.

Theorem (T_α is symmetric; two-sided Ward identity). *Let V be a Lorentzian regular ball and T_α the local Ward-fixed product of I-B-G-Q-B thm:Q-lorentzalpha(e), built from the defining data stated there: T_ν^F, the inverse Fourier transform of $-\mathrm{i}aP(k)[\log((q-\mathrm{i}0)/\nu^2) - c]$ with $P_{\mu\nu}(k) = \eta_{\mu\nu} q - k_\mu k_\nu$ (I-B-G-Q-B prop:Q-Ldeclared); Δ_F with $\mathscr{F}^{-1}[(q-\mathrm{i}0)^{-1}] = \mathrm{i}\Delta_F$; and $H_\alpha^L(x, x') = h_\alpha(u, \theta)/(8\pi^2 r r')$ (I-B-G-Q-B def:Q-HL).*

1. *$T_\nu^F(x - x')$, X_{cr}^F, X_{HH} and $s(x)\eta_{\mu\nu}\delta(x - x')$ are each invariant under* sw. *Hence* $\mathrm{sw}\, T_\alpha = T_\alpha$ *on* $V \times V$.

2. *Consequently T_α^{bv} is symmetric on $M^\circ \times M^\circ$, $\mathrm{T}(j_a j_b')$ of Theorem is symmetric under $(a, x) \leftrightarrow (b, x')$, and $\partial_{x'}^b \mathrm{T}(j_a j_b') = 0$: the Ward identity holds in both arguments.*

Proof. (a) T_ν^F. $P_{\mu\nu}(k) = P_{\nu\mu}(k) = P_{\mu\nu}(-k)$ and $q = k^2$ is even, so the symbol is symmetric in the indices and even in k. The inverse Fourier transform of an even tempered distribution is even. Hence $T^F_{\nu,\mu\nu}(y) = T^F_{\nu,\nu\mu}(-y)$, which is sw -invariance of $T_\nu^F(x - x')$.

The functions. $D(x,x') := \Delta_F(x-x')$ is symmetric, $D(x,x') = D(x',x)$, because $(q-\mathrm{i}0)^{-1}$ is even in k. $H := H^L_\alpha$ is symmetric: w and rr' are symmetric in (x,x'), the developed angle changes sign, $\theta \mapsto -\theta$, and $f_\alpha(u,\theta)$ and $f_1(u,\theta)$ depend on θ only through $\cos(\theta/\alpha)$ and $\cos\theta$. On $V \times V$, H is smooth (I-B-G-Q-B thm:Q-lorentzalpha(a); this smoothness is part of the definition of T_α and is not re-proved here), so the identity, valid off the set where the defining formula is singular, extends by continuity.

X^F_{cr} *and* X_{HH}. With $\partial = \partial_x$, $\partial' = \partial_{x'}$,

$$X_{\mathrm{cr},\mu\nu} = 2\,\partial_\mu D\,\partial'_\nu H + 2\,\partial_\mu H\,\partial'_\nu D - 2D\,\partial_\mu\partial'_\nu H - 2H\,\partial_\mu\partial'_\nu D.$$

Apply sw: exchange $x \leftrightarrow x'$ and $\mu \leftrightarrow \nu$. Since D and H are symmetric, $(\partial_\nu D)(x',x) = (\partial'_\nu D)(x,x')$ and likewise for H, and $\partial_\nu\partial'_\mu$ becomes $\partial'_\nu\partial_\mu$. The first two terms are exchanged with each other and the last two are mapped to themselves. Each product is a smooth function times a distribution, and pull-back by a diffeomorphism respects such products, so $\mathrm{sw}\,X_{\mathrm{cr}} = X_{\mathrm{cr}}$. The same computation with D replaced by H gives $\mathrm{sw}\,X_{HH} = X_{HH}$.

Seagull. $\eta_{\mu\nu}$ is symmetric and $s(x)\delta(x-x') = s(x')\delta(x-x')$.

(b) Proposition (b), whose hypothesis is now (a), gives the symmetry of T^{bv}_α; Proposition (a) gives that of T, and Proposition (c) the identity in x'. Check Q23 repeats the index computation of (a) symbolically with generic symmetric D and H.

par *Does not imply.* The theorem uses only the defining formulas of T_α as stated in the source record; the smoothness of H^L_α, which is part of what makes T_α defined at all, is the source's statement and is not re-proved. It does not construct a time-ordered product satisfying the Epstein–Glaser axioms, does not supply a state or unitarity for $\alpha \notin \{1/N\}$, does not identify T^{bv}_α with the inherited product, and admits no coefficient. No quantum coefficient is admitted. It does not imply a higher time-ordered product. par

Point-dependent diagonal terms

Let $A = C^\infty(M^\circ) \otimes \mathscr{W}$ (coefficients depending smoothly on x', valued in $\mathscr{W} = \mathscr{V} \otimes \mathbb{C}$) and $A[\xi]$ the polynomials in

$\xi_0, \dots, \xi_3$ with coefficients in A. Put

$$D^b := \xi^b - \partial^b_{x'} \quad \text{acting on} A[\xi].$$

The operators ξ^a (multiplication) and D^b commute with each other and among themselves.

Lemma (Koszul exactness for *D*). *Let $K^D_\bullet$ be the Koszul complex of $(D^0, \dots, D^3)$ on $A[\xi]$: $K^D_p = \Lambda^p A[\xi]^4$, differential contraction with D.*

1. *$K^D_\bullet$ is exact in degrees $p \geq 1$.*

2. *Every $f \in A[\xi]$ can be written uniquely as $f = D^b g_b + f_0$ with $f_0 \in A$ (of ξ-degree 0), modulo the choice of g; that is, $A[\xi]/\sum_b D^b A[\xi] \cong A$ via $f \mapsto f_0$. The same holds componentwise for tensors with values in a free A-module.*

Proof. Filter by ξ-degree. The top-degree part of $D \cdot z$ is $\xi \cdot z_n$, where z_n is the top part of z, because $\partial_{x'}$ does not raise the degree. (a) Let $z \in K^D_p$, $p \geq 1$, with $D \cdot z = 0$ and top degree n. Then $\xi \cdot z_n = 0$, so $z_n = \xi \cdot y$ with $y \in K_{p+1}$ homogeneous of degree $n - 1$, by exactness of the Koszul complex of the regular sequence ξ over A (as in the proof of Theorem (a)). Then $z - D \cdot y$ is a cycle of degree $\leq n - 1$. Descend to degree 0: a cycle z with constant-in-ξ entries has $\xi \cdot z = \partial \cdot z$, whose left side has degree 1 and right side degree 0, so $\xi \cdot z = 0$ and $z = 0$ for $p \geq 1$. Collecting the y's writes the original z as a boundary. (b) Existence: if f has top degree $n \geq 1$, $f_n = \xi^b g_b$ with g_b of degree $n - 1$, and $f - D^b g_b$ has degree $\leq n - 1$; descend. Uniqueness: if $f_0 = D^b g_b$ with f_0 of degree 0, take g of least top degree m. The degree-$(m + 1)$ part gives $\xi^b g_{b,m} = 0$, so $g_m = \xi \cdot k$ with $k \in K_2$, and $g - D \cdot k$ has the same image (since $D \cdot D \cdot k = 0$) and lower top degree, unless $g = 0$. So $f_0 = 0$.

par *Does not imply.* A standard filtration argument written out; no input is used. It does not imply anything about the physical choice of coefficients. par

Theorem (Two-sided classification with point-dependent coefficients). *Let $D_{ab} = \sum_\beta d^\beta_{ab}(x')\partial^\beta_x \delta(x - x')$ with $d^\beta_{ab} \in A$, finitely many β, and symbol $P_{ab}(x'; \xi) = \sum_\beta d^\beta_{ab}(x')\xi^\beta \in A[\xi]$. Then $\mathrm{T}(j_a j'_b) + D_{ab}$ satisfies the Ward identity in x and in x' if and only if*

$$P_{ab} = \xi^c(\xi^d - \partial^d_{x'})R_{cadb} + \xi^c\,\Theta_{cab}(x'),$$

where $R_{cadb} \in A[\xi]$ *satisfies* $R_{cadb} = -R_{acdb} = -R_{cabd}$, *and* $\Theta \in A \otimes \Lambda^3$ *is totally antisymmetric, independent of* ξ, *and divergence-free:* $\partial^b_{x'}\Theta_{cab} = 0$. *For constant coefficients this reduces to Theorem (a),(b).*

Proof. By Theorem (b), T itself satisfies both identities, so the conditions concern D alone. The x-identity is $\xi^a P_{ab} = 0$, and the x'-identity is $\xi^b P_{ab} - \partial^b_{x'} P_{ab} = 0$, i.e. $D^b P_{ab} = 0$ (proof of Theorem (b),(e)).

Sufficiency. $\xi^a\xi^c(\cdots)R_{cadb} = 0$ and $\xi^a\xi^c\Theta_{cab} = 0$ by antisymmetry in (c,a). Since the D's commute with each other and with ξ, $D^bD^dR_{cadb} = 0$ by antisymmetry in (d,b), and $D^b\xi^c\Theta_{cab} = \xi^c\xi^b\Theta_{cab} - \xi^c\partial^b_{x'}\Theta_{cab} = 0$.

Necessity. View $P \in K_1^\xi \otimes K_1^D$ (first index for ξ, second for D), with d_1 the ξ-contraction and d_2 the D-contraction; they commute. From $d_1P = 0$ and exactness of K^ξ at degree 1, $P = d_1\Lambda$ with $\Lambda_{ca;\,b}$ antisymmetric in (c,a). Then $d_1(d_2\Lambda) = d_2P = 0$, so $d_2\Lambda = d_1\Theta$ with $\Theta \in K_3^\xi \otimes K_0^D$. By Lemma (b), componentwise, $\Theta = d_2\Theta' + \Theta_0$ with $\Theta' \in K_3^\xi \otimes K_1^D$ and Θ_0 of ξ-degree 0, totally antisymmetric. Put $\Lambda' = \Lambda - d_1\Theta'$; then $d_1\Lambda' = P$ and $d_2\Lambda' = d_1\Theta_0$. Let $E_{ca;\,b} = \Theta_{0,\,cab}$. Then $d_2E = \xi^b\Theta_{0,\,cab} - \partial^b\Theta_{0,\,cab} = d_1\Theta_0 - \tau$ with $\tau_{ca} = \partial^b_{x'}\Theta_{0,\,cab}$, of degree 0. Hence $d_2(\Lambda' - E) = \tau$. So τ lies in the image of d_2 and has degree 0; by the uniqueness in Lemma (b), $\tau = 0$, which is the divergence condition. Then $\Lambda' - E$ is a d_2-cycle in degree 1, so by Lemma (a) $\Lambda' - E = d_2R$ with $R_{ca;\,db}$ antisymmetric in (c,a) and (d,b). Finally $P = d_1\Lambda' = d_1d_2R + d_1E$, which in indices is the displayed form. For constant coefficients the D's become ξ's on the relevant homogeneous parts and Θ is constant, which is Theorem (a).

par *Does not imply.* The theorem classifies admissible diagonal terms with point-dependent coefficients; it does not select one, does not fix the scale, and does not identify any with the inherited product. Operator-valued coefficients are covered only as elements of $\mathscr{W}$, as in version 5. No quantum coefficient is admitted. It does not imply a higher time-ordered product. par

Corollary (Lower degrees; consistency).

1. *With* $M = \deg P$, *the homogeneous parts satisfy Theorem (e) and in addition:* $P^0 = 0$*;*

$P^1 = \xi^c \Theta_{cab}$ + (degree-1part of$\xi^c D^d R_{cadb}$); *and the only constraint beyond the recursion is the divergence condition on* Θ. *Given any doubly transverse top part* P^M *with* $M \geq 2$, *a full admissible P with that top part exists (R homogeneous of degree* $M - 2$ *gives one).*

2. *Version 4's term* $Q(x')(\xi_a \xi_b - \eta_{ab}\xi^2)$ *is of the form* $\xi^c \xi^d R_{cadb}$ *with* $R_{cadb} = -Q(\eta_{cd}\eta_{ab} - \eta_{cb}\eta_{ad})$. *Its admissible completion is*

$$\xi^c(\xi^d - \partial^d_{x'})R_{cadb} = Q(\xi_a \xi_b - \eta_{ab}\xi^2) + \eta_{ab}\,\xi^c \partial_c Q - \xi_b\,\partial_a Q.$$

*For non-constant Q the degree-*1 *completion is nonzero. So the pure term* $Q(\xi_a \xi_b - \eta_{ab}\xi^2)$ *is not admissible for non-constant Q, as version 5's audit note says, but it becomes admissible after adding this first-order term.*

Proof. (a) P^0 is a constant-in-ξ tensor with $\xi^a P^0_{ab} = 0$, so $P^0 = 0$. The other statements are read off the displayed form, which has no ξ-degree-0 part. (b) $\xi^c \xi^d R_{cadb} = -Q(\xi^2 \eta_{ab} - \xi_a \xi_b)$, and $-\xi^c \partial^d R_{cadb} = \xi^c(\eta_{cd}\eta_{ab} - \eta_{cb}\eta_{ad})\partial^d Q = \eta_{ab}\,\xi \cdot \partial Q - \xi_b \partial_a Q$. Directly: ξ^a of the completion is $Q(\xi^2 \xi_b - \xi_b \xi^2) + \xi_b\,\xi \cdot \partial Q - \xi_b\,\xi \cdot \partial Q = 0$, and ξ^b minus $\partial^b_{x'}$ of it is $(\xi_a\,\xi \cdot \partial Q - \xi^2 \partial_a Q) - (\xi_a\,\xi \cdot \partial Q - \xi^2 \partial_a Q + \partial_a(\xi \cdot \partial Q) - \xi \cdot \partial\,\partial_a Q) = 0$. Check Q22 verifies the identities of Theorem for random point-dependent data.

par *Does not imply*. A reading of Theorem . It selects no term and admits no coefficient, and it does not imply a higher time-ordered product. par

Remark (Three or more currents, after version 6). Version 5's obstruction (Remark) is unchanged: the fully contracted three-current loop needs wavefront control of G^{bv}_{α} on all of M°, including at the diffracted front, and no input supplies it. Theorem removes one prerequisite (symmetry of the two-point building block), and Theorem classifies the two-point renormalization freedom, but neither gives the microlocal spectrum condition needed to multiply three boundary values in different invariants. Higher products stay open.

Audit of version 5. Theorem , Proposition and Remark were rechecked. No mathematical error was found.

Theorem (e) is correct as stated, as a necessary condition. Together with $\xi^a P^m_{ab} = 0$ for every m (not only $m = M$) its conditions are also sufficient, and Theorem solves them explicitly; the divergence condition on Θ is the case $m = 1$ of the recursion (V5-N1, scope note).

Routing note (not registered). Nothing has been registered, pushed or written to the site.

Version 7: wavefront positivity, and the three-current product

Version 7 answers request t249u (``work harder''). Remarks and left three-current products open because no input supplies wavefront control of G^{bv}_α on all of M°. Theorem (a) already contains that control in a weak form. Here it is sharpened, using analyticity in the angle (Lemma), and turned into a product criterion for every graph of contractions (Lemma). This gives a three-current product on all of $M^\circ \times M^\circ \times M^\circ$ with symmetry, time ordering, causal factorization and the Ward identity off the total diagonal (Theorem). The contact Ward identity on the total diagonal, and four or more currents, stay open (Remark).

Throughout, $w_{ij} = w(x_i, x_j)$ and θ_{ij} is the angle difference of x_i, x_j (both defined off the axis), and G_{ij} denotes $W^T_\alpha(x_i, x_j)$, which equals G^{bv}_α off the diagonal (Theorem (c)).

Lemma (Wavefront positivity).

1. *Let* $u(w,\theta) = \Gamma_\theta(w + \mathrm{i}0)$ *on* $\mathbb{R} \times \mathbb{R}$ *(with* θ *taken mod* $2\pi\alpha$*). Then* $\mathrm{WF}(u) \subseteq \{(w,\theta;\xi_w,\xi_\theta) : \xi_w > 0,\ |\xi_\theta| \le 3\xi_w\}$.

2. *At an off-diagonal pair* $(x, x') \in M^\circ \times M^\circ$*, every* $\Xi \in \mathrm{WF}(G^{\mathrm{bv}}_\alpha)_{(x,x')}$ *has the form* $\Xi = \xi_w\, \mathrm{d}w + \xi_\theta\, \mathrm{d}\theta$ *with* $\xi_w > 0$*; at such points* $\Delta t \neq 0$*. The same holds for every derivative of* G^{bv}_α.

3. *Over the diagonal,* $\mathrm{WF}(W^T_\alpha) \subseteq N^*\mathrm{diag} = \{(x,x;\xi,-\xi)\}$.

Proof. (a) $g_\alpha(u,\theta)$ is analytic in θ as well as in u, so $F(w,\theta) = \Gamma_\theta(w)$ extends holomorphically to complex (w,θ) off its singular set. By Lemma , Proposition and the closed form, near a real point that set is contained in $\{w = -1\}$ and in the finitely many curves $w = \cos(\theta + 2\pi\alpha k)$ that meet the strip $|\mathrm{Im}\, u| < \pi$. Since $|\mathrm{Im}\cos(a + \mathrm{i}b)| = |\sin a \sinh b| \le \sinh|b| < 2|b|$ for $|b| < 1$, none of these points lies in $\Gamma = \{\mathrm{Im}\, w > 3|\mathrm{Im}\,\theta|\}$, where

$|\operatorname{Im} w - \operatorname{Im} w_p| \geq \operatorname{Im} w/3$ for every singular w_p. Locally $F \cdot (w+1)\prod_k (w - \cos(\theta + 2\pi\alpha k))$ is bounded (poles of order at most one in each factor, with the $(w+1)^{-1/2}$ and shadow terms of Lemma (b) and Proposition (b)), so $|F| \leq C(\operatorname{Im} w)^{-N}$ on Γ near the point. By Hörmander's theorem on boundary values (*The Analysis of Linear Partial Differential Operators I*, Thm. 8.1.6; documentary record) the limit from Γ exists and its wavefront set lies in the dual cone $\Gamma^\circ = \{\xi : \xi_w \eta_w + \xi_\theta \eta_\theta \geq 0 \ \forall \eta \in \Gamma\} = \{\xi_w \geq |\xi_\theta|/3\}$. Approaching along $(\mathrm{i}\varepsilon, 0) \in \Gamma$ gives u. A nonzero $\xi \in \Gamma^\circ$ has $\xi_w > 0$.

(b) Off the diagonal, at points where u is singular, $\Delta t \neq 0$ (proof of Theorem (b): $\Delta t = 0$ forces $w \geq 1$, and $w = 1$ is singular only on the diagonal) and $\partial_t w = -\Delta t/(rr') \neq 0$, while θ does not depend on t. So $(x, x') \mapsto (w, \theta)$ is a submersion there and $G_\alpha^{\mathrm{bv}} = u(w,\theta)/(8\pi^2 rr')$. The pull-back theorem (Hörmander, Thm. 8.2.4; documentary) gives the form of Ξ. Elsewhere off the diagonal G_α^{bv} is smooth. Derivatives do not enlarge wavefront sets.

(c) On a regular ball $W_\alpha^T - \Delta_F - H_\alpha^L$ is supported on the diagonal (Theorem (c), Theorem (c)), hence is a finite sum of derivatives of $\delta(x - x')$ with smooth coefficients, whose wavefront set is in $N^*\mathrm{diag}$. H_α^L is smooth on the ball, and $\mathrm{WF}(\Delta_F)$ over the diagonal is $N^*\mathrm{diag}$.

par *Does not imply*. The lemma is a statement about the declared boundary value only. It does not imply the Hadamard property, positivity or any identification with an inherited propagator, and it says nothing at the axis.

par

Lemma (Product criterion for graphs of contractions). *Let $\mathcal{G}$ be a finite multigraph on $\{1, \dots, n\}$ without self-loops. To each edge $e = \{i, j\}$ attach a derivative of G_{ij} or of $\delta(x_i - x_j)$, and let $P_{\mathcal{G}}$ be the product of these distributions on $(M^\circ)^n$, times a smooth function. At a configuration x, let $\mathcal{G}_x$ be the subgraph of edges whose endpoints coincide. If $\mathcal{G}_x$ is a forest (in particular if it has no repeated edge), then $P_{\mathcal{G}}$ is well defined near x by Hörmander's product criterion (Thm. 8.2.10; documentary). In particular:*

1. *every $P_{\mathcal{G}}$ is defined on the pairwise-distinct configurations;*

2. *every $P_{\mathcal{G}}$ with $\mathcal{G}$ a forest is defined on all of $(M^\circ)^n$;*

3. *the triangle $G_{12}G_{23}G_{31}$ and its derivatives are defined off the total diagonal.*

Proof. Let $Y = \sum_k t_k \partial_{t_k}$. For an off-diagonal edge, $Y(w_{ij}) = -\Delta t_{ij}^2/(r_i r_j)$ and $Y(\theta_{ij}) = 0$ (check Q25), so by Lemma (b) every nonzero covector Ξ_e of that edge has $\Xi_e(Y) = \xi_w\, Y(w_{ij}) < 0$. For an edge with $x_i = x_j$, $\Xi_e = (\xi, -\xi)$ in the (x_i, x_j) slots (Lemma (c); for δ the same), so $\Xi_e(Y) = \xi_t(t_i - t_j) = 0$. Suppose $\sum_e \Xi_e = 0$ with each Ξ_e in the wavefront set of its factor or zero. Pairing with Y gives $\sum_{e \notin \mathscr{G}_x} \Xi_e(Y) = 0$, so every off-diagonal Ξ_e vanishes. The remaining Ξ_e, $e \in \mathscr{G}_x$, are conormal and sum to zero; if $\mathscr{G}_x$ is a forest, a leaf vertex meets one edge only, whose covector must vanish there, and induction removes all edges. So no zero sum of nonzero covectors exists, which is Hörmander's condition. (a): $\mathscr{G}_x = \varnothing$. (b): subgraphs of forests are forests. (c): off the total diagonal, at most one edge of the triangle has coincident ends.

par *Does not imply*. The lemma gives existence of unrenormalized products only. It does not imply that any cycle of coincident points (a loop at a diagonal) has a canonical extension, and it does not construct any time-ordered product by itself. par

The three-current product. Let $j_a = \mathrm{i}(\varphi^* \partial_a \varphi - \partial_a \varphi^* \varphi)$ (Wick ordered), $G = W_\alpha^T$, $c = \mathrm{i}$, and let $\mathrm{T}_2(j_a j_b')$ be the two-current product of Theorem . For a pair $\{i,j\} \subset \{1,2,3\}$ and the remaining index k, let $\mathscr{C}_G(X, j_k)$ denote the Wick expansion of a Wick polynomial X in (x_i, x_j) against $j(x_k)$, with every contraction between them replaced by G. Define, as a form on $\mathscr{D} \times \mathscr{D}$,

$$\mathrm{T}_3(j_a j_b j_c) = [\sum_{\mathscr{G}} P_{\mathscr{G}}]_{\text{no cycle}} + \sum_{\{i,j\}} T_\alpha^{\mathrm{bv}}(x_i, x_j)\,\mathrm{mathopen}:j(x_k)\mathrm{mathclose}: + \sum_{\{i,j\}} \mathscr{C}_G(2c\,\eta\,\delta(x_i - x_j)\mathrm{mathopen}:\varphi^*\varphi\mathrm{mathclose}:(x_i), j_k)_{\text{open}},$$

where the first sum runs over all Wick contraction patterns of $j_a j_b j_c$ whose graph has no cycle (normal-ordered term, one contraction, chains), the second replaces the double edge $\{i,j\}$ (the two-current loop) by v2's global product, and in the third only the terms with at most one contraction are kept. The two fully contracted triangles and the fully contracted seagull terms are omitted; part (c) below shows that this is forced.

Theorem (Three-current product; partial). *Let* $0 < \alpha \le 1$.

1. *Existence.* T_3 *is a well-defined form with values in* $\mathscr{D}'((M^\circ)^3)$*, on all of* $(M^\circ)^3$*, including every diagonal.*

2. *Symmetry.* T_3 *is invariant under every permutation of* $(a,x_1),(b,x_2),(c,x_3)$.

3. *Furry cancellation. Off the total diagonal, the sum of the two fully contracted triangles vanishes identically (check Q24), and every fully contracted seagull term vanishes identically. Under the declared requirement of charge-conjugation covariance (*$\varphi \leftrightarrow \varphi^*, j \mapsto -j$*, under which* ω_α *is invariant), the c-number part of any extension of* T_3 *to the total diagonal vanishes. So omitting these terms loses nothing.*

4. *Time ordering. On each open set* $\{t_{\sigma 1} > t_{\sigma 2} > t_{\sigma 3}\}$ *of pairwise-distinct configurations,* T_3 *equals the Wick expansion in* ω_α *of the operator product* $j(x_{\sigma 1})j(x_{\sigma 2})j(x_{\sigma 3})$.

5. *Causal factorization. On* $\{t_k > \max(t_i,t_j)\}$*, including the partial diagonal* $x_i = x_j$*,* $\mathrm{T}_3 = \mathscr{C}_G(\mathrm{T}_2(j_i j_j), j_k)$ *minus its fully contracted part, which vanishes there by (c); likewise for* $\{t_k < \min(t_i,t_j)\}$.

6. *Ward identity off the total diagonal.* $\partial^a_{x_1}\mathrm{T}_3(j_a j_b j_c) = 0$ *on* $(M^\circ)^3$ *minus* $\{x_1 = x_2 = x_3\}$*, and likewise in each variable.*

Proof. (a) Every acyclic pattern is a forest, so Lemma (b) applies, with the smooth matrix elements of Lemma as smooth factors. $T^{\mathrm{bv}}_\alpha(x_i,x_j)$ is a distribution on $(M^\circ)^2$ (Theorem (a)), tensored with a smooth function of x_k. The kept seagull terms are forests with one δ-edge and at most one G-edge.

(b) The set of patterns is permutation invariant, G and T^{bv}_α are symmetric (Theorem (a), Theorem), and the seagull sum runs over all pairs.

(c) Reversing the orientation of a triangle exchanges the roles of φ and φ^* at each vertex. Since ω_α is quasifree with $\langle\varphi\varphi^*\rangle = \langle\varphi^*\varphi\rangle = W$, the propagators are unchanged, and since j is antisymmetric under the exchange, the term changes by $(-1)^3$. The two triangles are defined off the total diagonal (Lemma (c)), so their sum vanishes there. A fully contracted seagull term is $\delta(x_i - x_j)$ times $\mathrm{i}[G\,\partial G - \partial G\,G](x_i,x_k) = 0$. Under charge conjugation $\mathrm{T}_3 \mapsto (-1)^3\mathrm{T}_3$; covariance and invariance of ω_α force the c-number part to equal its negative.

(d) There $G_{ij} = W_\alpha(x_i,x_j)$ when $t_i > t_j$ (Theorem (b)), the double-edge terms equal $\mathrm{B}[W_\alpha]$ (as in the proof of Theorem (b)), and seagull terms vanish; Wick's theorem for the quasifree ω_α gives the operator product, whose

triangle part is zero by (c).

(e) Expand $\mathscr{C}_G(\mathrm{T}_2(j_i j_j), j_k)$ using the four terms of T_2: the normal-ordered term gives the patterns without an $\{i,j\}$ edge; S_{ij} gives those with one $\{i,j\}$ edge, including the triangles; the seagull gives the seagull terms; $T^{\mathrm{bv}}_\alpha 1$ gives the double-edge term. All contractions with x_k are $W_\alpha(x_k,\cdot) = G$ there, so this is the Wick expansion of $j(x_k)\mathrm{T}_2(j_i j_j)$.

(f) Take a point off the total diagonal; there are three cases. *Case* $x_1 \notin \{x_2, x_3\}$ (possibly $x_2 = x_3$). By the Leibniz rule (valid for Hörmander products), $\partial^a_{x_1}$ of each pattern acts on the two slots of x_1. In $\mathrm{i}(\psi_1^* \partial_a \psi_2 - \partial_a \psi_1^* \psi_2)$ the cross terms cancel and the rest is $\mathrm{i}(\psi_1^* \mathrm{Box}\, \psi_2 - \mathrm{Box}\, \psi_1^* \psi_2)$. Each slot function ψ is a contraction $G(x_1, x_l)$ with $x_l \neq x_1$ or a field matrix element, and $\mathrm{Box}\, \psi = 0$ near the point (Theorem (d), Lemma). The terms $T^{\mathrm{bv}}_\alpha(x_2, x_3)\mathrm{mathopen}: j(x_1)\mathrm{mathclose}:$ and $T^{\mathrm{bv}}_\alpha(x_1, x_l)\mathrm{mathopen}: j(x_m)\mathrm{mathclose}:$ are divergence-free by current conservation and Theorem (b). Seagull terms with a δ at x_1 vanish near the point. In the seagull term at $\{2,3\}$, the slot at x_1 is a $G(x_1,\cdot)$ or an open field. *Case* $x_1 = x_2 \neq x_3$. Near the point, $\mathrm{T}_3 = \mathscr{C}_G(\mathrm{T}_2(j_1 j_2), j_3)$ minus the triangles, which sum to zero there by (c). This is the term-by-term comparison of (e), which uses no time condition: the double edges $\{1,3\}$ and $\{2,3\}$ of $\mathscr{C}_G$ are $G^2 = \mathrm{B}[G] = T^{\mathrm{bv}}_\alpha$ near points with $x_3 \neq x_1, x_2$. Differentiation in x_1 commutes with $\mathscr{C}_G$, because the contraction of $\partial\varphi(x_1)$ with $\varphi^*(x_3)$ is $\partial_{x_1} G(x_1, x_3)$. The proof of Theorem (c) is an identity of Wick polynomials in (x_1, x_2) with distribution coefficients. It uses only $\mathrm{Box}_x G = c\,\delta$, the field equation of the open slots, and the algebraic identity $(N_{b'} - N_b)|_{\mathrm{diag}} = 0$. After contraction with $j(x_3)$, an open slot becomes $G(\cdot, x_3)$, which satisfies the field equation near the point, and the algebraic identity survives because contraction is linear. So $\partial^a_{x_1} \mathrm{T}_3 = \mathscr{C}_G(\partial^a_{x_1} \mathrm{T}_2(j_1 j_2), j_3) = 0$ near the point. *Case* $x_1 = x_3 \neq x_2$. The same with 2 and 3 exchanged, using (b).

par *Does not imply*. The theorem is a free-field, form-level, three-current statement on the off-axis region, and it is *partial*. It does not imply the Ward identity on the total diagonal (the contact identity), does not classify the diagonal-supported freedom of T_3, and constructs no product of four or more currents and no interacting product. Charge-conjugation covariance is a declared requirement, not an inherited one. It does not prove the Hadamard property, does not identify ω_α or T_3 with inherited objects, does not verify (TC), and leaves RG open. No quantum

coefficient is admitted. par

Remark (What stays open after version 7). (i) *Contact Ward identity.* $\partial_{x_1}^{a} \mathrm{T}_3$ is a distribution supported on the total diagonal. Whether it vanishes, or can be removed by adding diagonal-supported operator terms, is not decided here; this is the three-point analogue of Theorem (c) and needs the scaling degree of the forest terms at the total diagonal. (ii) *Four or more currents.* Lemma defines every pattern off the configurations where coincident points carry a cycle. For $n \geq 4$, loops through some but not all points (for example a two-current loop inside a four-current product, or a square on three coincident points) must be renormalized at partial diagonals, and the extension, its symmetry and its Ward identity are not constructed. (iii) Furry's argument kills all fully contracted odd loops off the total diagonal for every odd n, by the same sign count.

Audit of version 6. Theorem , Lemma , Theorem , Corollary and Remark were rechecked. No mathematical error was found. One scope note (V6-N1): Remark (and Remark) say that the wavefront control needed for the three-current loop is ``not supplied by any input''. The orientation statement of Theorem (a), together with analyticity of g_{α} in the angle, does supply it off the total diagonal (Lemma). What no input supplies is control at coincident points, which is the renormalization question of Remark .

Routing note (not registered). Nothing has been registered, pushed or written to the site.

Version 8: the contact Ward identity, four currents, and non-inheritance

Version 8 answers request t253u (plan section 5). Version 7 left open the Ward identity of T_3 on the total diagonal and all products of four or more currents (Remark). Here the contact identity is proved, at the form level of Theorem (Theorem). The plan anticipated a contact contribution of classified Koszul form with a coefficient computed from the residue of G_{α}^{bv}; instead the contact contribution is zero, and no new coefficient enters. A four-current product is constructed off the total diagonal with the Ward identity there, and its only obstruction, three kinds of C-even cycles at the total diagonal, is located (Proposition , partial). Proposition is extended to every (T0) datum without a hypersurface piece in its singular support (Proposition).

Standing ledger of plan t253u (verbatim). No quantum coefficient is admitted. Host A and Host B stay separate; Host C is not created. O.1 stays NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved); no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 and ST4b stay NOT_SUPPLIED. The last reading bars stay open. Continuum acquisition stays a negative certificate. No Einstein dynamics, derived metric, spacetime emergence, Page curve, or quantized relational time without an external clock is claimed. Citations identify source records only; no proof transfer. No compilation is asserted. All expansions are documentary separations or permanent negative certificates.

(The ledger's ``T3" is the standing item of that name, not the three-current product T_3 of this sheet. On the last ledger sentence, see plan correction PC21.)

Theorem (Contact Ward identity for three currents). *Let* $0 < \alpha \le 1$. *The three-current product* T_3 *of Theorem satisfies* $\partial^{a}_{x_1} \mathrm{T}_3(j_a j_b j_c) = 0$ *on all of* $(M^{\circ})^3$*, including the total diagonal, as a form on* $\mathscr{D} \times \mathscr{D}$*; likewise in each variable. No diagonal-supported term has to be added, and the only coefficients that enter are those of Theorem : the operator seagull* $2c$ *and its c-number value* $s = 2ch$*, where* h *is fixed by the residue data of Theorem (c).*

Proof. Split T_3 into its c-number part, which is 0 (Theorem (c)), and its operator part. The operator part is a finite sum of three kinds of term. (i) $T^{\mathrm{bv}}_{\alpha}(x_i, x_j)$mathopen : $j(x_k)$mathclose :. (ii) Acyclic contraction patterns with at least one open field. (iii) Seagull terms $2c\,\eta\,\delta(x_i - x_j)$ times mathopen : $\varphi^{*}\varphi$mathclose : (x_i), with at most one contraction to $j(x_k)$. Every term of kind (ii) or (iii) is a forest product (a δ-edge counts as an edge), so it is defined on all of $(M^{\circ})^3$ (Lemma (b)).

Terms of kind (i). ∂_{x_1} of $T^{\mathrm{bv}}_{\alpha}(x_1, x_l)$mathopen : $j(x_k)$mathclose : vanishes by Theorem (b), and that of $T^{\mathrm{bv}}_{\alpha}(x_2, x_3)$mathopen : $j(x_1)$mathclose : by current conservation (Lemma).

Terms of kinds (ii) and (iii). All products involved, and all products obtained from them by differentiating a factor, are defined on $(M^{\circ})^3$, so the Leibniz rule holds globally (continuity of the Hörmander product). As in the proof of

Theorem (f), the divergence in x_1 of a pattern equals $\mathrm{i}(\psi_1^* \mathrm{Box}\,\psi_2 - \mathrm{Box}\,\psi_1^*\psi_2)$ summed over its two slot functions at x_1. An open slot gives $\mathrm{Box}\,\psi = 0$. A contracted slot $G(x_1, x_l)$ gives $\mathrm{Box}_{x_1} G = c\,\delta(x_1 - x_l)$, now on all of $M^\circ \times M^\circ$ (Theorem). The resulting terms are $c\,\delta(x_1 - x_l)$ times the rest of the pattern. The rest has at least one open field, so at most one further contraction runs from $\{x_1, x_l\}$ to x_k, and each such term is again a forest product. Group the terms by $l \in \{2, 3\}$. For fixed l, the terms of kind (ii) with a contraction between x_1 and x_l, together with the seagull terms at $\{1, l\}$, are exactly the terms S_{1l} and $2c\,\eta\,\delta\mathrm{mathopen}:\varphi^*\varphi\mathrm{mathclose}:$ of $\mathrm{T}_2(j_1 j_l)$, with the open slots either left open or contracted once with $j(x_k)$. The computation in the proof of Theorem (c) uses three ingredients:

• $\mathrm{Box}\,G = c\delta$;

• $\delta(x - x')F = \delta\,F|_{\mathrm{diag}}$ and the corresponding rule for $\partial\delta$;

• the identity $(N_{b'} - N_b)|_{\mathrm{diag}} = 0$ of Wick polynomials. After at most one contraction with $j(x_k)$ the first holds as stated. The second holds because the remaining factors are transversal to $\{x_1 = x_l\}$ (the conormal of $\{x_1 = x_l\}$ does not meet the wavefront set of $G(x_1, x_k)$ or $G(x_l, x_k)$ pulled back to $(M^\circ)^3$, as in Lemma). The third survives because contraction is linear. So these terms cancel for each l, exactly as $\partial^a S_{ab}$ cancels against the seagull there. One family of δ-terms remains. It comes from the seagull at the pair $\{2, 3\}$, $2c\,\eta\,\delta(x_2 - x_3)\mathrm{mathopen}:\varphi^*\varphi\mathrm{mathclose}:(x_2)$, with one slot contracted to $j(x_1)$. In it, $\mathrm{Box}_{x_1} G(x_1, x_2) = c\,\delta(x_1 - x_2)$ gives two terms, one from each choice of contracted slot at x_1. Their sum is

$$\mathrm{i}c\,\delta(x_1 - x_2)\,[\mathrm{mathopen}:\varphi^*(x_1)\varphi(x_2)\mathrm{mathclose}: - \mathrm{mathopen}:\varphi^*(x_2)\varphi(x_1)\mathrm{mathclose}:] \cdot 2c\,\eta\,\delta(x_2 - x_3).$$

The propagator is the same for both orderings (Theorem (c)), and on the support of $\delta(x_1 - x_2)$ the bracket restricts to $\mathrm{mathopen}:\varphi^*\varphi\mathrm{mathclose}: - \mathrm{mathopen}:\varphi^*\varphi\mathrm{mathclose}: = 0$. This is the same algebraic identity as $(N_{b'} - N_b)|_{\mathrm{diag}} = 0$. The patterns without a contraction at x_1 have divergence 0 (open slots only). Hence $\partial^a_{x_1} \mathrm{T}_3 = 0$ on $(M^\circ)^3$. The other variables follow by symmetry (Theorem (b)).

par *Does not imply.* The theorem is a free-field, form-level identity on the off-axis region. It does not imply that T_3 is the unique Ward-preserving three-current product: the diagonal-supported freedom compatible with the identity

(the analogue of Theorems , and) is not classified. It constructs no interacting product, does not verify (TC) or the IX-E-3 trace certificate, does not identify ω_{α} with an inherited state, and leaves RG open. No quantum coefficient is admitted. par

Proposition (Four currents off the total diagonal; partial). *Define* $\mathrm{T}_4(j_a j_b j_c j_d)$ *as the sum of all Wick contraction patterns of the four currents and of the seagull vertices* $2c\,\eta\,\delta(x_i - x_j)\mathrm{mathopen}:\varphi^{*}\varphi\mathrm{mathclose}:(x_i)$, *in every admissible combination: zero, one or two seagulls on disjoint pairs. Make the following replacements:*

• *a double edge between two current vertices becomes* $T_{\alpha}^{\mathrm{bv}}(x_i, x_j)$*;*

• *a triangle of current vertices becomes* 0*;*

• *a double edge between a seagull vertex and a current vertex becomes* 0.

1. *Existence. The remaining terms are of two kinds. Forest products, and current double edges tensored with forest products, are defined on all of* $(M^{\circ})^4$. *The c-number cycles are defined on* $(M^{\circ})^4 \setminus \Delta_4$, *where* $\Delta_4 = \{x_1 = x_2 = x_3 = x_4\}$. *There are three kinds of c-number cycle: the two* 4*-cycles of currents (``squares"); the seagull loops* $\delta(x_i - x_j)\,\Pi(x_i; x_k, x_l)$, *a closed loop through the seagull vertex and both other currents; and the seagull bubbles* $\delta(x_i - x_j)\delta(x_k - x_l)\,G(x_i, x_k)^2$.

2. *Furry. The replaced triangles and seagull–current double edges sum to zero over the two orientations off their own diagonals. The squares, seagull loops and seagull bubbles are C-even, so Furry's argument does not remove them.*

3. *Ward identity off* Δ_4. $\partial^{a}_{x_1}\mathrm{T}_4 = 0$ *on* $(M^{\circ})^4 \setminus \Delta_4$, *and likewise in each variable.*

4. *Open. An extension of the C-even cycles across* Δ_4 *that keeps the Ward identity is not constructed.*

Proof. (a) Each current vertex has two slots, and so does each seagull vertex. A pattern is therefore a disjoint union of paths and cycles in the graph whose vertices are the currents and seagulls (check Q27 enumerates all patterns without seagulls). A cycle using only current vertices is a double edge, a triangle or a square. A double edge exhausts the slots of its two vertices, so the rest of the pattern lives on the other variables. For a cycle through

a seagull vertex at $\{i, j\}$, the δ identifies $x_i = x_j$. The cycle's coincident subgraph is then a cycle only if all of its vertices coincide, and with the δ-edges this means Δ_4. Two exceptions remain: the seagull–current double edge, at $x_i = x_k$, is replaced by 0; and a seagull loop with one current open is impossible, because the loop uses all slots. For squares, the coincident subgraph at x is a path unless $x \in \Delta_4$. Forests are covered by Lemma (b). (b) The sign count of Theorem (c) gives $(-1)^{\#\text{current vertices on the cycle}}$, because mathopen : $\varphi^{*}\varphi$mathclose : is C-even. A seagull–current double edge is $\delta \cdot \mathrm{i}[G\partial G - \partial G\, G] = 0$, as in Theorem (c). (c) Fix a point off Δ_4 and split the indices into clusters of coincident points; there are at least two clusters. Near the point, every contraction between different clusters is G, a smooth function there. Comparing term by term as in the proof of Theorem (e), T_4 equals $\mathscr{C}_G$ of the products $\mathrm{T}_{|A|}$ of the clusters ($\mathrm{T}_1 = j$, T_2 of Theorem , T_3 of Theorem), up to two kinds of term:

• triangles spanning two clusters, which the comparison produces but T_4 omits; their two orientations cancel there by (b);

• inter-cluster current double edges, where $\mathrm{B}[G^2] = T^{\mathrm{bv}}_{\alpha}$ off the diagonal. Differentiation in x_1 commutes with $\mathscr{C}_G$ (proof of Theorem (f)). It annihilates the cluster product containing x_1, by Theorem (c) or Theorem , applied as identities of Wick polynomials with distribution coefficients. Hence $\partial_{x_1} \mathrm{T}_4 = 0$ near the point. (d) This is open; see Remark .

par *Does not imply*. The proposition constructs T_4 off Δ_4 only, and it is *partial*. It does not imply that the C-even cycles have a Ward-preserving extension across Δ_4. It does not compute their scaling degree, and it says nothing about five or more currents or about interacting products. No quantum coefficient is admitted. par

Proposition (Non-inheritance for every (T0) datum without a hypersurface piece). *Let u° satisfy (T0) of I-B-G-Q-B thm:Q-notinherited on $\mathbb{R}^4 \setminus W$, with W of dimension two, and suppose that* sing supp u° *contains no nonempty relatively open subset of a smooth hypersurface. Then no extension of u° equals $\Phi^{*} T^{\mathrm{bv}}_{\alpha}(\cdot, x')$ for a diffeomorphism Φ and an off-axis x'. This holds for every $0 < \alpha \le 1$, and for each image cone and each piece of the diffracted front with $A_\theta \neq 0$ separately (Propositions and), not only for the direct cone.*

Proof. An extension u of u° satisfies sing supp $u \subseteq$ sing supp $u^\circ \cup W$. On a regular ball about x', $T^{\mathrm{bv}}_\alpha(\cdot, x')$ is singular on the direct light cone of x' minus the vertex, a smooth hypersurface Σ (proof of Proposition (b)). If $u = \Phi^* T^{\mathrm{bv}}_\alpha(\cdot, x')$, then $\Phi^{-1}(\Sigma)$ is a smooth hypersurface contained in sing supp $u^\circ \cup W$. Its relatively open subset $\Phi^{-1}(\Sigma) \setminus W$ is nonempty, because W has dimension two, and it lies in sing supp u°. That contradicts the hypothesis. For an image cone the singular part is $1/(4\pi^2(\sigma_k + \mathrm{i}0))$ with multiplicity $m_k \geq 1$ (Proposition (c)), again a hypersurface singularity. On the diffracted front the singular part is $A_\theta(w)(w + 1 + \mathrm{i}0)^{-1/2}$, singular wherever $A_\theta \neq 0$ (Lemma (b)); the shadow terms of Proposition add further poles and remove none. Each gives the same argument.

par *Does not imply*. Documentary: this is a singular-support separation. It does not imply that a datum whose singular support contains a hypersurface piece *is* realized by T^{bv}_α (Proposition (b) shows that the test is then silent). It does not identify T^{bv}_α with any inherited datum. par

Remark (What stays open after version 8). (i) The diagonal-supported freedom of T_3 compatible with Theorem . (ii) The extension of the C-even cycles of T_4 (squares, seagull loops, seagull bubbles) across Δ_4 with the Ward identity, and five or more currents. (iii) The IX-E-3 trace certificate (TC), which stays unverified; the Hadamard property on M°; identification with inherited data; RG.

Audit of version 7. Lemma , Lemma , Theorem and Remark were rechecked. No error in any proof or theorem statement was found. Remark contains two scope errors. (V7-N1) Remark (i) says the contact identity ``needs the scaling degree of the forest terms". It does not: forest products are defined across the total diagonal, and the identity follows algebraically (Theorem ). Scaling degrees matter only for cycles at the total diagonal, which first appear in $\mathrm{T}_4$. (V7-E1) Remark (ii) lists ``a square on three coincident points" as needing renormalization at a partial diagonal. On three coincident points, the coincident subgraph of a square is a path (two edges). That subgraph is a forest, so the square is defined near such points by the criterion of Lemma ; the only cycle singularity of a square is at Δ_4 (Proposition (a)). Likewise, the ``two-current loop inside a four-current product" named there is a double edge and is already defined by T^{bv}_α. Remark itself was correct to leave four currents open.

Routing note (not registered). Nothing has been registered, pushed or written to the site.

Release

Theorem (I-B-G-Q-C-v2 release). *For* $0 < \alpha \leq 1$, *under the declared boundary value (Declaration):*

1. G_α^{bv} *is a global off-diagonal solution of the wave equation. It is the flat Feynman image term on each image cone, counted with multiplicity, and it has an inverse-square-root front with a selected branch, with an added pole at shadow angles (Theorem ; v3 correction, Proposition).*

2. *The Ward-fixed product globalizes to a conserved* T_α^{bv}. *Its computed representative is* $R_\alpha = \mathrm{i}(\alpha^{-2} - 1)/(24\pi^2 r^2)\,\eta\,\delta$ *, (Theorem). The conjugate prescription forces* $-R_\alpha$ *(v3 correction, Proposition).*

3. R_α *is separated from the inherited datum by type. For* T_α^{bv} *the separation holds for the benchmark only (Proposition).*

4. *The support bridge for the same-prescription kernel holds (Theorem).*

5. *(v2)* G_α^{bv} *is the time-ordered two-point function of the static ground state of the Friedrichs dynamics, which is positive and has unitary time evolution. The opposite prescription is the anti-time ordering. Off the diagonal* T_α^{bv} *is the ground-state time-ordered bilinear (Theorem , Corollary).*

6. *(v3) The two-current time-ordered product of the complex free field in* ω_α *exists as a form on smooth finite-particle vectors. It satisfies causal factorization off the diagonal and the operator Ward identity with total seagull* $2\mathrm{i}\eta_{ab}\delta(\mathrm{mathopen} : \varphi^* \varphi \mathrm{mathclose} : +h)$ *(Theorem). This is partial.*

7. *(v4) The shadow set is determined exactly (Proposition). Diagonal terms compatible with the operator Ward identity are classified by the Koszul form of their symbols; order-zero terms are excluded, and the order-two family is two-sided compatible only for constant coefficients (Theorem).*

8. *(v5) The diagonal terms with point-independent coefficients that preserve both Ward identities are exactly the doubly transverse symbols* $\xi^c \xi^d R_{cadb}$ *and* $\xi^c \Theta_{cab}$*; the Lorentz-covariant c-number ones are* $(\partial_a \partial_b - \eta_{ab}\mathrm{Box})S(\mathrm{Box})\delta$

(Theorem). Symmetry of $\mathrm{T}(j_a j'_b)$ *is equivalent to symmetry of the local product (Proposition).*

9. *(v6)* T_α *is symmetric, so* $\mathrm{T}(j_a j'_b)$ *is symmetric and conserved in both arguments (Theorem). The diagonal terms with point-dependent coefficients compatible with both identities are exactly* $\xi^c(\xi^d - \partial^d_{x'})R_{cadb} + \xi^c \Theta_{cab}$, Θ *a divergence-free 3-form (Theorem).*

10. *(v8) The three-current product satisfies the Ward identity on all of* $(M^\circ)^3$, *with no added diagonal term (Theorem). The four-current product exists, with the Ward identity, off the total diagonal (Proposition).*

11. *(v7) Off the diagonal,* $\mathrm{WF}(G^{\mathrm{bv}}_\alpha)$ *has positive* d*w component (Lemma); every acyclic contraction pattern is a distribution on all configurations (Lemma); the three-current product exists globally, is symmetric and causally factorized, has vanishing c-number part, and satisfies the Ward identity off the total diagonal (Theorem ; partial).*

Proof. Collect the cited results.

par *Does not imply*. Beyond item (5), which is a two-point statement for the Friedrichs free field, the release does not imply a composite time-ordered product, an inherited state, a selected extension or state, the inherited coefficient, a verified (TC), a scale, or any change of the standing ledger. The certificate stays RG-open. par

Release box (I-B-G-Q-C historical v3; book file I-B-G-Q-C-v2). Version 3 of a new sheet; it revises no other sheet, and v1 and v2 are retained read-only. The boundary-value prescription is declared as new data.

• *Proved:* the global boundary value; the globally conserved product, with forced representative $R_\alpha = \mathrm{i}\frac{\alpha^{-2}-1}{24\pi^2 r^{-2}}\eta_{\mu\nu}\delta$; the conjugate prescription forces $-R_\alpha$; the support bridge for the closed completion relation; the ground-state two-point identification; and (v3) the two-current time-ordered product at form level, with operator Ward identity and forced operator seagull, which is *partial*.

• *Corrected in v3:* the residue multiplicity at $\theta = 0$; the shadow-angle poles; and the conjugate contact.

• *Open:* higher time-ordered products and the full Epstein–Glaser axioms; the Hadamard property on M°; inherited identification and C^γ; the IX-E-3 certificate (TC); and RG. No quantum coefficient is admitted. Host C is not

created; Host A and Host B stay separate. O.1 is NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 stays NOT_SUPPLIED; thm:J is not substituted. The last reading bars stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal, quantized relational time without an external clock, or Page curve is claimed. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only. Authorship requires confirmation. Nothing is pushed, registered or written to the site. No independent review has been performed.

Release box (I-B-G-Q-C historical v4; book file I-B-G-Q-C-v2). Version 4 of this sheet; it revises no other sheet, and v1–v3 are retained read-only.

• *Proved in v4:* the exact shadow set and its multiplicities; the classification of diagonal terms compatible with the operator Ward identity (Koszul form; no order-zero term; the order-two family two-sided only for constants).

• *Open:* higher time-ordered products and the full Epstein–Glaser axioms; the two-sided and scaling-degree normalization of diagonal terms in general; the Hadamard property on M°; inherited identification and $C^{\vee}$; the IX-E-3 certificate (TC); and RG. No quantum coefficient is admitted. Host C is not created; Host A and Host B stay separate. O.1 is NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 stays NOT_SUPPLIED; thm:J is not substituted. The last reading bars stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal or Page curve is claimed. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only; no proof transfer. Authorship requires confirmation. Nothing is pushed, registered or written to the site. No independent review has been performed.

Release box (I-B-G-Q-C historical v5; book file I-B-G-Q-C-v2). Version 5 of this sheet; it revises no other sheet, and v1–v4 are retained read-only. *Proved in v5:* the two-sided Ward classification of diagonal terms with

point-independent coefficients, the Lorentz-covariant count $\lfloor p/2 \rfloor$, and the leading-symbol condition for point-dependent coefficients (Theorem); the reduction of symmetry of the two-current product to the local product (Proposition). Checks Q20, Q21 are computer-assisted confirmations, not proof steps; no independent review has been performed. *Open:* higher time-ordered products and the full Epstein–Glaser axioms (Remark); the two-sided classification for point-dependent operator-valued coefficients beyond the leading symbol; symmetry of the local product. No quantum coefficient is admitted. Host C is not created; Host A and Host B stay separate. O.1 is NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 stays NOT_SUPPLIED; thm:J is not substituted. The last reading bars stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal or Page curve is claimed. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only; no proof transfer. Authorship requires confirmation. Nothing is pushed, registered or written to the site. No independent review has been performed.

Release box (I-B-G-Q-C historical v6; book file I-B-G-Q-C-v2). Version 6 of this sheet; it revises no other sheet, and v1–v5 are retained read-only. *Proved in v6:* symmetry of the local product T_{α} from its defining formula, hence symmetry and the two-sided Ward identity of $\mathrm{T}(j_a j_b^{'})$ (Theorem); the classification of diagonal terms with point-dependent coefficients (Theorem). *Exact checks:* Q22, Q23. *Open:* products of three or more currents and the full Epstein–Glaser axioms (Remark); inherited identification; RG. No independent review has been performed. No quantum coefficient is admitted. Host C is not created; Host A and Host B stay separate. O.1 is NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 stays NOT_SUPPLIED; thm:J is not substituted. The last reading bars stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal or Page curve is claimed. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only; no proof transfer. Authorship requires confirmation. Nothing is pushed, registered or written to the site. No independent

review has been performed.

Release box (I-B-G-Q-C historical v7; book file I-B-G-Q-C-v2). Version 7 of this sheet; it revises no other sheet, and v1–v6 are retained read-only.

• *Proved in v7:* wavefront positivity of the declared boundary value (Lemma); a product criterion for contraction graphs (Lemma); a global three-current product with symmetry, time ordering, causal factorization, Furry cancellation and the Ward identity off the total diagonal (Theorem).

• *Partial:* the three-current row. *Open:* the contact Ward identity on the total diagonal; four or more currents; Hadamard property on M°; identification; (TC); RG. Audit of v6: no error; scope note V6-N1. No independent review has been performed. No quantum coefficient is admitted. Host C is not created; Host A and Host B stay separate. O.1 is NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 stays NOT_SUPPLIED; thm:J is not substituted. The last reading bars stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal or Page curve is claimed. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only; no proof transfer. Authorship requires confirmation. Nothing is pushed, registered or written to the site. No independent review has been performed.

Release box (I-B-G-Q-C historical v8; book file I-B-G-Q-C-v2). Version 8 of this sheet; it revises no other sheet, and v1–v7 are retained read-only.

• *Proved in v8:* the contact Ward identity of T_3 on all of $(M^{\circ})^3$, with no new coefficient (Theorem); non-inheritance for every (T0) datum without a hypersurface piece in its singular support (Proposition).

• *Partial:* four currents, defined with the Ward identity off the total diagonal (Proposition). *Open:* the diagonal freedom of T_3; the C-even cycles of T_4 at Δ_4; five or more currents; Hadamard property on M°; identification; the IX-E-3 trace certificate (TC), which stays unverified; RG. Audit of v7: no error in proofs or statements; scope

errors V7-N1 and V7-E1 in Remark . No independent review has been performed. No quantum coefficient is admitted. Host C is not created; Host A and Host B stay separate. O.1 is NOT_CLOSED. XVI-N is not discharged (N-Ins.4 unresolved), and there is no Route-B successor. XV-B is not opened. H2, I3 and T3 stay unresolved; J2 stays Unresolved. U6-sel is not moved. Res02-A Q1–Q4 and A1–A2 stay open. Gate 0 stays NOT_SUPPLIED; thm:J is not substituted. The last reading bars stay open. No Einstein dynamics, derived metric, spacetime emergence, external-clock removal or Page curve is claimed. Continuum acquisition stays a negative certificate. No compilation is asserted. Citations identify source records only; no proof transfer. Authorship requires confirmation. Nothing is pushed, registered or written to the site. No independent review has been performed.

appendix

Open table

begingroup toprule Open item Status

midrule endhead Selection of the prescription (state, time ordering) v2: w + i0 is the time ordering and w − i0 the anti-time ordering of ω_α (Theorem (d)). The choice of extension and state (ω_α versus thermal or other) is open

(v2) TO product / state for $\alpha \notin \{1/N\}$ two-point level: **obtained** (Theorem). v3: two-current product at form level **obtained, partial** (Theorem); higher composite products and full Epstein–Glaser axioms: not obtained. Hadamard property beyond the regular balls: open

(v2) IX-E-3 (TC) for the inherited operator not verified

(v2) RG status open

Actual inherited lift and its C^γ not constructed; representative of (OD) lifts only

Identification with the inherited datum under (T0) alone, global product undecided (Proposition (b))

IX-E-3 (TC) for $L = \mathrm{E}_0 C$ open, as in IX-E-3

$\alpha > 1$ (angle excess) not treated

RG status RG-open

Standing ledger items unchanged (Section)

(v3) Operator-valued diagonal terms with derivatives of δ **v4: classified** relative to the Ward identity in x (Theorem); two-sided compatibility decided for the order-two family $Q(\partial_a\partial_b - \eta_{ab}\mathrm{Box})\delta$ only

(v4) Two-sided Ward classification of all diagonal terms; scaling-degree and symmetry normalization **v5: closed for point-independent coefficients** (Theorem (a)–(d)), with the Lorentz-covariant count $\lfloor p/2 \rfloor$ under the order bound; **partial** for point-dependent coefficients (leading symbol, Theorem (e))

(v5) Symmetry of $\mathrm{T}(j_a j'_b)$ on the diagonal **v6: proved** (Theorem); v5: reduced to the local product

(v5) Two-sided classification with point-dependent coefficients **v6: closed** (Theorem)

(v5) Three-current and higher products open: needs wavefront information on G^{bv}_{α} (Remarks ,). **v7: partial**: wavefront information supplied off the diagonal (Lemma); three-current product global, symmetric, causally factorized, Ward identity off the total diagonal (Theorem); contact Ward identity and $n \geq 4$ open (Remark)

(v7) Contact Ward identity of T_3 **v8: proved** (Theorem); diagonal freedom compatible with it not classified

(v7) Four or more currents **v8: partial**: T_4 off Δ_4 with the Ward identity (Proposition); C-even cycles at Δ_4 and $n \geq 5$ open

(v4) Location of shadow angles **determined** (Proposition)

(v3) Causal relation of M° versus the closed completion relation in Theorem the theorem is stated only for the latter (Remark)

bottomrule endgroup

Separation and edge table

begingroup toprule Sheet Labels Edge

midrule endhead I-B-G thm:green, thm:closed, thm:transfer Friedrichs Green operators on the cone; they do not supply the boundary value, which is declared.

I-B-VP-2 lem:boundaryvalue, eq:BVorientation, prop:criterion, lem:diffnormal The boundary-value lemma is used. prop:criterion and lem:diffnormal concern the free covariance and are not needed; no interacting product is taken from them.

I-B-G-Q thm:compatibility, thm:dini, thm:fourier, prop:v5-codim2, prop:v5-ward Criteria only; the benchmark of prop:v5-codim2(iii) is used in Proposition (c).

I-B-G-Q-B def:Q-Galpha, lem:Q-Galpha, def:Q-cone, thm:Q-lorentzalpha, thm:Q-global, cor:Q-globalobs, thm:Q-actualdefect, thm:Q-conecount, thm:Q-notinherited, thm:Q-282, thm:Q-run, thm:Q-product Closed form, local product and the method obstruction: the obstruction is answered by declared data, not removed. The singularity type at the front is corrected (Section).

IX-E, IX-E-3 thm:obstruction, thm:algebraic, thm:geometric, thm:surface-exact, thm:minimal, eq:sphere-ball, eq:imagK Different kernel (Remark); no transfer.

VII-H thm:release Free susceptibility; not identified.

Oai282-A, Oai282-B ass:ward, thm:ward, thm:classification Comparison only (authors OpenAI); not premises.

bottomrule endgroup

Plan corrections

begingroup toprule # Plan item Correction

midrule endhead PC1 ``declare Friedrichs boundary value from I-B-VP-2 lem:boundaryvalue" lem:boundaryvalue is a one-variable boundary-value theorem with parameters; it does not name a prescription. The declared datum is the side $\operatorname{Im} w > 0$ (Declaration). The Friedrichs choice enters through G_{α} (regular modes), not through the lemma.

PC2 thm:Q-282 ``(seagull $s = 2\mathrm{i}H$ global on diagonal; diffracted residue vanishes iff $\alpha = 1/N$; every agreeing lift

has defect class zero)" These are cor:Q-globalobs(a), thm:Q-global(b) and thm:Q-actualdefect(a). thm:Q-282 is the formal Oai282-A transcription (Remark).

PC3 NB ``running does not place the product outside that finite Ward certificate" Agrees with thm:Q-282 and the v5 summary; no conflict. It also holds for the global product.

PC4 ``continue seagull off diagonal" The seagull is diagonal-supported. What continues is its coefficient function $2\mathrm{i}H_{\alpha}$, as $2\mathrm{i}H_{\alpha}^{\mathrm{bv}}$ (Theorem (e)). The representative itself is unchanged.

PC5 ``test vs thm:Q-notinherited (point vs codim-2); expect identification to fail" It fails for the representative. For the global Lorentzian product the point test is not available (its slice singular support is three-dimensional); it fails only for the benchmark (Proposition).

PC7 (v2) t233u: ``Negative theorem: decl:bv prescription does not produce a state/TO product for $\alpha \notin \{1/N\}$ (unitarity/positivity not supplied)" **false**. The prescription is the time ordering of the Friedrichs ground state, which is positive and unitary (Theorem). What is true is that version 1 did not supply this, and that composite TO products are not constructed

PC8 (v2) t233u open row ``TO product/state for $\alpha \notin \{1/N\}$ – not obtained" replaced by the row ``two-point level obtained; composite not obtained"

PC9 (v2) t233u: ``global product's singular-support test is inconclusive under (T0) alone" correct; sharpened by Proposition , which separates every datum whose singular support has no hypersurface piece

PC10 (v2) t233u: ``IX-E-3 certificate unverified for its own kernel" the IX-E-3 certificate (TC) concerns the inherited operator $L = \mathrm{E}_0 C$; it is unverified, as in version 1. The IX-E-3 kernel IX-E-3 eq:imagK is handled there by geometry (Remark)

PC6 Source availability All nine sources resolved on the site (HTTP 200). All named labels exist.

PC11 t240u: ``composite Wick-current time-ordered product" Done in part: the two-current product at form level, with an operator Ward identity and a forced operator seagull (Theorem). Higher products and the full

Epstein–Glaser axioms are not done.

PC12 t240u (v2 wording): ``residue exactly 1" Holds only for $\theta \neq 0$. At $\theta = 0$ the residue is 2 (Proposition (a)).

PC13 t240u (v2 wording): ``inverse square root, not a pole in w" Fails at shadow angles (Proposition (b)).

PC14 t240u (v2 wording): ``representative unchanged by the prescription" False: the conjugate prescription forces $-R_{\alpha}$ (Proposition).

PC15 t240u: ``J is the causal relation" It is the closed completion relation (Remark).

PC16 t245u: ``work harder" on the v3 open rows The diagonal-term row is settled relative to the Ward identity (Theorem). Higher products, Hadamard, identification, (TC) and RG are not attempted beyond v3 and stay open.

PC17 v3 reading ``shadows occur for $\alpha = 1/N$" Only for even N (Proposition (c)).

PC18 t246u: ``I-B-G-Q-C-v2 TT product" The two-current product is advanced (two-sided classification, symmetry reduction). No higher product is constructed, and the full Epstein–Glaser axioms are not claimed; that row stays open.

PC19 t247u: ``symmetry of the I-B-G-Q-B local product, lower-degree point-dependent classification, three-current products" Symmetry: proved from the defining formula (Theorem). Point-dependent classification: closed (Theorem). Three-current products: not constructed; the obstruction is unchanged (Remark).

PC20 t249u: ``I-B-G-Q-C-v2: three-or-more-current products (symmetry / Ward identity / obstruction)" Three currents: symmetry and Ward identity off the total diagonal proved, with Furry cancellation (Theorem); the contact identity is not decided, so the row is partial. Four or more: the obstruction is identified (renormalization of sub-loops at partial diagonals, Remark); not constructed.

PC21 (v8) t253u ledger: ``All expansions are documentary separations or permanent negative certificates" quoted verbatim in Section . Theorem and Proposition are theorems on this sheet's own declared objects (free field, declared boundary value). They are neither, and they move no ledger item. The ledger's ``T3" is the standing item, not T_{3}

PC22 (v8) t253u: ``contact contribution has the classified Koszul form; compute its coefficient from the residue of G_{α}^{bv}'' the contact contribution is *zero* (Theorem), trivially of Koszul form. No new coefficient arises; the only coefficients are v3's seagull $2c$ and $s = 2ch$, with h from the residue data (Theorem (c)). The labels thm:diag4, thm:two5, thm:pt6 exist; they classify diagonal *freedom*, which is not needed for the identity and is not classified for T_3

PC23 (v8) t253u: ``Thm bridge for singular support'' and ``EM fibre constancy'' misdescribed. Theorem controls the *support* of $\operatorname{Im} G_{\alpha}^{\mathrm{bv}}$ (inside J), not its singular support; singular support is Lemma and Theorem . EM-OB-1-v5 thm:EM-fibre exists, but it is an elementary factorization criterion and plays no role in four-current products; it is not used

PC24 (v8) t253u: ``Lem wf7 / Lem graph7 transversality for all contracted graphs; partial (contact open)''; ``Prop notinh strengthened via Prop shadow4 + Prop mult3'' correct and carried out. Graph products are defined off coincident cycles (Lemma), and for four currents the cycles at the total diagonal are the obstruction (Proposition). Proposition is the strengthening; shadow4 and mult3 add further hypersurface pieces. All labels named (diag4, two5, pt6, wf7, graph7, notinh, shadow4, mult3, bridge) exist

bottomrule endgroup

Errors

Audit of v7 (v8). No error was found in the proofs or theorem statements of v7. Remark contains two scope errors. (V7-N1) Item (i) says the contact identity needs scaling degrees of forest terms; it does not (Theorem). (V7-E1) Item (ii) lists ``a square on three coincident points'' as needing renormalization at a partial diagonal. There the coincident subgraph is a path, so the square is defined by Lemma ; squares are singular only at $\Delta_4$. Its ``two-current loop'' is a double edge, already defined by T_{α}^{bv}.

Audit of v6 (v7). No mathematical error was found in v6. (V6-N1, scope note) Remarks and say that the wavefront control needed for the three-current loop is not supplied by any input; off the total diagonal it follows

from Theorem (a) and analyticity in the angle (Lemma). The missing control is at coincident points only.

Audit of v5 (v6). No mathematical error was found in v5. (V5-N1, scope note) Theorem (e) is a correct necessary condition; together with $\xi^a P^m_{ab} = 0$ for all m it is also sufficient, and Theorem solves it. Proposition 's open hypothesis is discharged by Theorem .

Audit of v4 (v5). No error was found in v4. Theorem (d) is recovered as a special case of Theorem (c),(e).

Audit of v3 (v4). No mathematical error was found in v3. (V3-N1, imprecision) Proposition (b) says ``Shadows occur for $\alpha = 1/N$, for example $\alpha = \frac{1}{2}$ and $\theta = 0$''. They occur only for even N, and then only at $\theta = 0$; for odd N there is none (Proposition (c)). The example given is correct. Theorem (d) is correct as stated and is extended by Theorem (b).

Errors in v2 (corrected in v3). (V2-E1) Lemma (a) said every pole has residue exactly 1. At $\theta = 0$ the images $\pm k$ coincide, and the residue is 2. For example, $\alpha = \frac{1}{3}$, $\theta = 0$ gives $\Gamma_0 = \frac{1}{w-1} + \frac{2}{w+1/2}$ (Q13). The per-image subtraction of Theorem (e) and Theorem (e) fails on $S_k \cap S_{-k}$. (V2-E2) Lemma (b) and the front formula of Theorem (e) fail at shadow angles $|\theta + 2\pi\alpha k| = \pi$, where Γ_θ has a pole $\frac{m}{2}(w+1)^{-1}$. For example, $\alpha = \frac{1}{2}$, $\theta = 0$ gives $\frac{1}{w-1} + \frac{1}{w+1}$ (Q13, Q15). (V2-E3) Theorem (d) and the abstract said that the opposite prescription has the same R_α. It has $-R_\alpha$ (Proposition). The proof of (d) confused the location of the singular set with the value of the forced diagonal term. (V2-E4, wording) ``$J = \{\sigma \le 0\}$ is the causal relation of the cone'' holds only for the completed cone (Remark). The distribution-level existence statements are unaffected, because the bounds of Lemma (d) survive (Proposition (c)).

v3 process notes. (E-v3-1) Check Q14 first extracted the residue as $-i\varepsilon\Gamma(w_\nu + i\varepsilon)$, with the wrong sign, and reported -2. With the sign fixed, the result was limited by precision ($2 + 1.4 \times 10^{-8}$, and garbage at $\varepsilon = 10^{-20}$). It was replaced by a contour integral on $|w - w_\nu| = 10^{-3}$, which is valid because Γ_θ is holomorphic on $\mathbb{C} \setminus (-\infty, -1]$ apart from its poles.

Errors in sources. (S1) I-B-G-Q-B thm:Q-global states that for $\alpha \notin \{1/N\}$, H^L_α ``has in addition a simple pole on

the diffracted front $w = -1$". I-B-G-Q-B cor:Q-globalobs(b) repeats that it has ``simple poles in w" there, with Sokhotski–Plemelj jump $2\pi i\,\delta$. The pole at $u = i\pi$ is simple in u. But $w + 1 = -\zeta^2/2 + \dots$ with $\zeta = u - i\pi$, so in w the singularity is $(w + 1)^{-1/2}$, which is locally integrable (Lemma (b); check Q3: $(w + 1)h \to 0$ while $\sqrt{2(w + 1)}\, h \to i\rho$). The two one-sided boundary values differ by an L^1_{loc} function on $\{w < -1\}$, not by a delta. The conclusion of cor:Q-globalobs(b), that a prescription is needed, still stands: the branches differ, and B[G] contains $(w + 1)^{-3}$. The image-cone statements are correct, and their residue in w is exactly 1. The source is not edited.

Version 1 statements superseded in v2. Version 1 recorded ``no time-ordered product, no state and no unitarity are supplied for $\alpha \notin \{1/N\}$". That was an accurate statement of what version 1 proved, but it is not a limitation of the prescription (Theorem ). The source non-claim of I-B-G-Q-B lem:Q-Galpha (``no positivity is claimed for $\alpha \notin \{1/N\}$") is consistent with this sheet: positivity is proved here, not claimed there.

Version 2 process errors (fixed). (E4) The first draft of Lemma (b) contained a garbled intermediate display, which was removed. (E5) The first numerical check of Lemma at complex ζ used 20-digit adaptive quadrature and did not finish in 750s. It was redone in double precision with fixed truncation $k \le 80/\mathrm{Re}\,\zeta$ (check Q12).

Process errors (fixed). (E1) The first check of the coincidence value used the invariant $w = \cosh u$ at $u = 10^{-8}$, and the cancellation lost precision; it was redone in u at 60 digits. (E2) The first wave-equation check used the angular term $r^{-2}\alpha^{-2}\partial_\psi^2$, which belongs to the undeveloped angle; with the developed angle ψ the correct term is $r^{-2}\partial_\psi^2$. (E3) The first front check for $\alpha = \frac{1}{2}$ used the wrong acceptance test.

Relation to the user drafts

The user drafts are C-D1 (sha256 de674836d5b27cf6c9c27de4655c92b2c30b266af8cddd85d9f0485133b122cf, ``Consolidated Closure Package", author ``Formal Closure Record"), C-D2 (sha256 ee8aaeae2dce9dac3fd7f9bcbd31c1f1119fbce00d1ee61c3ff90f51cd46dc31, ``Scoped Closures, Explicit Counterexamples, and the Remaining Gates"), C-D3 (sha256 c75da6f24cd8ead7e85c76095379cb2b15267c2ed9aab021b4bffcaea4d094b1, ``Further-Closed Consolidated

Ledger for Plans 1–14", added at t228u) and C-D4 (sha256 6b5a6e69fa5045412193432987a872884272993b5bcf83cdbc962120303b3c49, ``Plans 1–14: Further Mathematical Closures …", empty author, added at t229u). C-D3, C-D4 and some C-D2 items have no labels; those are cited by section title. All four were copied read-only and not edited. Only their Plan 4 claims are assessed here, and each was checked independently. begingroup toprule Claim Verdict Reason

midrule endhead C-D1 decl:Plan4: the prescription as a heat-kernel limit $e^{-\varepsilon A_F}$ ``of I-B-VP-2 lem:boundaryvalue" **mis-descrip-tion** lem:boundaryvalue is a one-variable holomorphic boundary-value theorem (PC1). A spatial heat-kernel regularization is a different limit, and the draft does not show that it exists for the products.

C-D1 thm:4-seagull(i), C-D1 eq:Fr-rep **refused** It writes $s(x,y) = 2\mathrm{i}H_F(x,y)\delta(x,y)$ as ``the global diagonal seagull of thm:Q-global"; thm:Q-global has no seagull. The representative contains an undefined $\mathscr{R}_\alpha^{\mathrm{diff}}$overleftrightarrow ∂overleftrightarrow $\partial\delta_\Sigma$ and an unexplained $c_\alpha(v)$. The computed representative is Theorem (c): R_α, with no worldsheet term.

C-D1 thm:4-seagull(ii): failure ``across the codimension-8 aligned-null locus", and thm:Q-notinherited ``proves that no point-diagonal or codimension-2 distribution can intertwine" it **false** thm:Q-notinherited is a codimension-two singular-support statement; it mentions no codimension-eight locus. The correct test is Proposition .

C-D1 thm:4-seagull(iii): running stays inside the Ward certificate conclusion correct As I-B-G-Q-B thm:Q-282; extended to the global product in Theorem (f).

C-D1 thm:4-IXbridge: (SB) holds on the flat realization under $\mathscr{B}_{\mathrm{Fr}}$ flat conclusion already in the source; proof refused IX-E-3 proves the flat bridge with its own kernel IX-E-3 eq:imagK, which is not a Friedrichs/Feynman kernel (Remark). The draft's heat-kernel step is truncated and unsupported. The same-prescription bridge is Theorem .

C-D2 prop:bv-product: opposite-side boundary values have no product limit correct; not an obstruction here All

factors in B[G_α^{bv}] are same-side boundary values, whose product is the boundary value of the product (Theorem (a)).

C-D2 Lemma ``Sphere–ball intersection'' ($t - s \le R \le t + s$) correct Lemma

C-D2 prop:ix-geometry correct As IX-E-3 thm:geometric

C-D2 Corollary ``Conditional analytic bridge'' correct (conditional) For the same-prescription cone kernel the support condition is *proved*, in the sharper form $\operatorname{supp}\operatorname{Im} \subseteq J$ (Theorem).

C-D2 S ``Exact ending'': representative and same-prescription bridge unresolved; ``no explicit off-diagonal seagull kernel … supplied'' superseded Under the declared datum both are computed (Theorems , ,). The closed form I-B-G-Q-B def:Q-Galpha was supplied.

C-D3 S ``Plan 4'', Proposition ``Sphere–ball intersection'' with $|t - s| \le |x - y| \le t + s$ **false** Counterexample $t = 1$, $s = 3$, $x = y$ (Lemma ; check Q7b). The correct lower condition is $t - s \le R$.

C-D3 Proposition ``Geometric exclusion'' $|x - y| > t + s$ correct Its proof needs the corrected lemma; the statement is IX-E-3 thm:geometric.

C-D3 status ``analytic bridge conditional on same-prescription kernel support'' superseded Proved for the declared kernel (Theorem).

C-D4 correction 1 ($t - s \le |x - y|$) and Theorem ``Correct intersection condition'', with Example correct; adopted Lemma . C-D4 is right and C-D3 is wrong.

C-D4 Theorem ``Flat geometric exclusion'' correct As IX-E-3 thm:geometric

C-D4: kernel support ``remains a separate analytic hypothesis'' superseded for the same-prescription kernel Theorem . For the IX-E kernel itself the hypothesis fails in the form $\operatorname{supp}\operatorname{Im} K \subseteq J$, because of the reflected slab; IX-E-3 handles that kernel by geometry.

C-D4 S ``An auxiliary product …'', Theorem ``Explicit representative, Ward identity, and running'' correct Re-derived (check Q8). This is the flat product of I-B-G-Q-B thm:Q-product and I-B-G-Q-B thm:Q-run with

general c_{0}; it is not new, and it is not the conical object.

C-D4 *missing*: ``native conical seagull continuation, same-prescription bridge … absent" superseded (except identification) The continuation and the bridge are computed here under the declared datum. The identification theorem is still absent, and Proposition decides only the benchmark.

C-D4 boundary (RG-open; no admission) correct Agrees

bottomrule endgroup None of the four drafts' Plan-4 sections is used as a premise without re-proof.

par *Version 2.* Request t233u supplied no further draft. In the light of Theorem , the C-D1 idea of a Friedrichs prescription is consistent with the declared datum: the declared boundary value is the Friedrichs ground-state time ordering. The verdict on its heat-kernel derivation is unchanged.

The t240u draft E-D3

E-D3 (sha256 db2d0dd898e8babe8a3672c1d00f82f12b77f66bf0d596c001f1dcad25d40968) was copied read-only to inputs/E-D3.tex. Its section ``I-B-G-Q-C-v2: spectral two-point certificate and Ward contact" and the relevant correction rows are assessed here. begingroup toprule Claim Verdict Reason

midrule endhead ``Sobolev and inverse-power control"; ``Ground-state two-point construction"; ``The declared boundary value" correct They agree with Lemma and Theorem .

``Image and diffracted singularities: multiplicity matters" correct; adopted Proposition (a), with the exact examples (Q13, Q14).

Shadow exception to the generic front formula correct; adopted and sharpened The pole coefficient $c_{\theta} = m/2$ is computed (Proposition (b), Q15).

``Ward contact for the plus prescription" correct It agrees with Theorem (c).

``Correction for the conjugate prescription" correct; adopted Proposition . Uniqueness makes $-R_{\alpha}$ forced, not only natural.

``Same-prescription support bridge" and the missing-axis qualification correct; adopted Remark .

Absence of a composite time-ordered product superseded in part The two-current product is constructed at form level (Theorem).

Ledger wording; ``Completion" framing replaced; refused The standing phrase is used verbatim.

bottomrule endgroup

Version 7. Request t249u supplied no new user draft. The verdicts above are unchanged.

Version 8. Request t253u supplied no new user draft. The verdicts above are unchanged.

Version 6. Request t247u supplied no new user draft. The verdicts above are unchanged.

Version 5. Request t246u supplied no new user draft. The verdicts above are unchanged.

Version 4. Request t245u supplied no new user draft. The verdicts above are unchanged.

Label audit

Every cited label of another sheet or of a user draft was found as a verb || in the read-only copy listed in Appendix (automated; 37 labels, 0 missing). A draft label being found means only that the cited claim exists, not that it is adopted (Appendix). Labels that the plan names but that do not exist are recorded as plan corrections. begingroup toprule Sheet Label Found

midrule endhead C-D1 decl:Plan4 yes

C-D1 eq:Fr-rep yes

C-D1 thm:4-IXbridge yes

C-D1 thm:4-seagull yes

C-D2 prop:bv-product yes

C-D2 prop:ix-geometry yes

I-B-G prop:nonGH yes

I-B-G thm:closed yes

I-B-G thm:green yes

I-B-G thm:transfer yes

I-B-G-Q prop:v5-codim2 yes

I-B-G-Q-B cor:Q-globalobs yes

I-B-G-Q-B def:Q-Galpha yes

I-B-G-Q-B def:Q-HL yes

I-B-G-Q-B def:Q-cone yes

I-B-G-Q-B lem:Q-Galpha yes

I-B-G-Q-B prop:Q-Ldeclared yes

I-B-G-Q-B thm:Q-282 yes

I-B-G-Q-B thm:Q-actualdefect yes

I-B-G-Q-B thm:Q-conecount yes

I-B-G-Q-B thm:Q-global yes

I-B-G-Q-B thm:Q-lorentzalpha yes

I-B-G-Q-B thm:Q-notinherited yes

I-B-G-Q-B thm:Q-product yes

I-B-G-Q-B thm:Q-run yes

I-B-VP-2 eq:BVorientation yes

I-B-VP-2 lem:boundaryvalue yes

IX-E thm:algebraic yes

IX-E thm:obstruction yes

IX-E-3 eq:imagK yes

IX-E-3 eq:sphere-ball yes

IX-E-3 thm:geometric yes

IX-E-3 thm:minimal yes

Oai282-A ass:ward yes

Oai282-A thm:ward yes

Oai282-B thm:classification yes

VII-H thm:release yes

bottomrule endgroup

Computer-assisted checks

All checks in this appendix are computer-assisted (floating point, mpmath, sympy, scipy). They corroborate the proofs, and no proof depends on them. Script code/checks.py, output logs/checks.json. begingroup toprule Id Content Result

midrule endhead Q1 h(0,0) alpha=0.7 OK (1.1e-13)

Q1 h(0,0) alpha=0.707107 OK (5.68e-14)

Q1 h(0,0) alpha=0.333333 OK (5.53e-14)

Q2 Res_w at image k=-1 alpha=0.45 theta=-0.2 OK ((0.999999999942 - 1.0e-13j))

Q2 Res_w at image k=1 alpha=0.45 theta=-0.2 OK ((1.0 - 9.99999999965e-14j))

Q3 front alpha=0.7 OK (sqrt(2(w+1))h= (1.23024 + 0.0j) ; pred= (1.2302599 + 0.0j) ; |(w+1)h|= 8.7e-6)

Q3 front alpha=0.5 OK (sqrt(2(w+1))h= (-0.00031663735 + 0.0j) ; pred= (-3.8833e-30 + 0.0j) ; |(w+1)h|= 2.24e-9)

Q4a real on spacelike pairs alpha=0.7 OK (1.0e-26)

Q4b Im G beyond front (w=-1.5) alpha=0.7 OK (Im(+i0)=-0.516254 Im(-i0)=0.516254)

Q4a real on spacelike pairs alpha=0.5 OK (1.0e-26)

Q4b Im G beyond front (w=-1.5) alpha=0.5 OK (Im(+i0)=-3.78983e-30 Im(-i0)=3.78983e-30)

Q5 image poles w_k<=cos(theta) OK (alpha in {0.3, 0.7, 0.999}, 41 theta, |k|<=6)

Q6 box G=0 at a global timelike pair (w=-1.1654) OK (|box G|=1.36e-31 |G|=0.02066)

Q7a sphere-ball: intersect iff t-s<=R<=t+s OK (3000 random triples; distance to sphere |R-t|)

Q7b counterexample to |t-s|<=R (C-D3) OK (t=1, s=3, R=0: sphere inside ball, |t-s|=2>R=0)

Q8 C-D4 auxiliary: u=r\{}-4 off 0, off-origin Pi, dA=0 OK (sympy)

Q9 Bessel identity int qJ0 I_nu K_nu = e\{}{-nu u}/(2ab sinh u), 5 orders x 2 configs OK (max rel err 5.93e-14)

Q10 Gamma(conj w)=conj Gamma(w) (anti-time ordering), 6 points OK (0)

Q11 reflection positivity of G_alpha (alpha=0.7, 10 random points): min eigenvalue >= -1e-20*max OK (min 0.005778 max 1.277)

Q12 Friedrichs spectral kernel (2B)\{}{-1}e\{}{-B zeta} (Hankel/J_nu modes) equals closed form at 3 complex zeta OK (max rel err 5.34e-12)

Q13 theta=0 partial fractions: alpha=1/3 -> 1/(w-1)+2/(w+1/2); alpha=1/2 -> 1/(w-1)+1/(w+1) OK (alpha=1/3: 4/(2*w + 1) + 1/(w - 1); alpha=1/2: 1/(w + 1) + 1/(w - 1))

Q14 theta=0, alpha=0.3: residue at w=cos(2 pi alpha) is 2 OK (residue (2.0 + 3.44015167e-34j))

Q15 shadow alpha=0.7, theta=-0.4pi: lim (w+1)Gamma(w) at w=-1 equals 1/2 OK ((w+1)Gamma at w+1=1e-6 i, 1e-8 i: (0.49991848 - 8.2139152e-5j), (0.49999185 - 8.1577879e-6j))

Q16 divergence of the one-contraction term: d\{}a S_ab = -[box G N_b' - (box G)_b' N + G_b' box N - G (box N)_b'] OK (symbolic identity, generic G, N, both b)

Q17 Prop shadow4: shadow set = {+-(pi-2 pi alpha k)} iff 1/alpha not odd (exact, 246 rational alpha); alpha=0.37, theta_s: (w+1)Gamma -> 1/2 OK ((w+1)Gamma at w+1=1e-9 i: (0.5000158 + 1.5799409e-5j))

Q18 Thm diag4(c): Ward-compatible degree-q symbols in 4D, dims [0, 6, 20, 45, 84] = 4C(q+3,3)-C(q+4,3); Koszul image spans the kernel (q=0..4, exact) OK (Koszul exactness confirmed in degrees <=4)

Q19 Thm diag4(d): x-Ward symbol identity holds; two-sided condition forces grad Q = 0 (exact) OK (solution dQ=0 only)

Q20 v5 Thm two5(a): doubly transverse degree-q 2-tensors in 4D, dims [0, 4, 35, 96, 196]; image of xi\{}c xi\{}d R_[ca][db] (q>=2) and of xi\{}c Theta_cab (q=1) has the same dims (exact rank, q=0..4) OK (kernel [0, 4, 35, 96, 196] image [0, 4, 35, 96, 196])

Q21 v5 Thm two5(c),(d): (xi_a xi_b - eta_ab xi\{}2) is transverse in a and in b and satisfies P_ab(xi)=P_ba(-xi) (exact, sympy) OK (checks (True, True, True))

Q22 v6 Thm pt6 / Cor pt6(b) (exact, sympy): random point-dependent R (antisymmetric pairs) and divergence-free Theta from a 4-form: xi\{}a P_ab=0, (xi\{}b-d\{}b)P_ab=0, div Theta=0; completion of Q(xi_a xi_b-eta xi\{}2) admissible; pure term with non-constant Q not admissible OK (checks (True, True, True, True, True, True))

Q23 v6 Thm sym6(a) (exact, sympy): X_cr and X_HH built from explicit symmetric D(x,y), H(x,y) are invariant under (mu,x)<->(nu,y), all 16 index pairs OK (ok=True)

Q24 v7 Thm T37(c) (exact, sympy): the two fully contracted triangles of j j j (j=i(phi* d phi - d phi* phi)), with a generic symmetric polynomial propagator in two dimensions and vertex-dependent derivative directions, sum to zero; 2 orientations OK (sum expands to 0)

Q25 v7 Lemma graph7 (exact, sympy): Y(w)=-(dt)\{}2/(r r') for Y=t d_t+t' d_t', d_t' w=(t-t')/(r r'), and dt=0 gives

w=1+((r-r')\{}2+dz\{}2)/(2rr')>=1 OK (identities hold)

Q26 v7 Lemma wf7(a) (numerical corroboration, mpmath): |Im cos(a+ib)|<=sinh|b|, and |Gamma_theta(w)|*(Im w)\{}2 stays bounded for complex theta with Im w>3|Im theta| (alpha=0.7) OK (max |F|(Im w)\{}2=0.113 over 3348 samples)

Q27 v8 Prop T48(a), Thm ward8 (exact, combinatorial): all 18 Wick patterns of 3 currents and 108 of 4 currents; cycle lengths [2, 3] and [2, 3, 4]; for every pattern and every set partition of the points, the coincident subgraph is a forest iff no cycle lies inside one cluster (so squares are singular only at the total diagonal) OK (n3=18 n4=108)

Q28 v8 Prop T48(b) (exact, sympy): with a generic symmetric polynomial propagator in two dimensions, the seagull-current double edge sums to 0 over orientations, while the seagull loop through two currents does not (C-even) OK (bubble=0, loop!=0)

bottomrule endgroup Result: 36 of 36 checks OK.

Artifact hashes (sha256)

begingroup toprule File sha256

midrule endhead inputs/C-D1.tex de674836d5b27cf6c9c27de4655c92b2c30b266af8cddd85d9f0485133b122cf

inputs/C-D2.tex ee8aaeae2dce9dac3fd7f9bcbd31c1f1119fbce00d1ee61c3ff90f51cd46dc31

inputs/C-D3.tex c75da6f24cd8ead7e85c76095379cb2b15267c2ed9aab021b4bffcaea4d094b1

inputs/C-D4.tex 6b5a6e69fa5045412193432987a872884272993b5bcf83cdbc962120303b3c49

inputs/E-D3.tex db2d0dd898e8babe8a3672c1d00f82f12b77f66bf0d596c001f1dcad25d40968

inputs/I-B-G-Q-B.api.tex 200bdfcc78685cbc46bf9e960914fac313125da146dc12fa702f22de6ede8767

inputs/I-B-G-Q-C-v1.tex 96b71a0d067fd3a18a8de4e7694aea16b3abcaa0077819da8b8a510ae8aea4cf

inputs/I-B-G-Q-C-v2.tex 68d0c2c0ed48dff07cd650b375acde936720d3524360ad228d11142f9719b92d

inputs/I-B-G-Q-C-v3.tex d71665d01f0cc3f72454e994e4369a64e1c697f5936c83e113b329d5b94cec43

inputs/I-B-G-Q-C-v4.tex eac51c7dba521e222434ee929522d6a2f61ad001c6ee729deaee36239237dfc1

inputs/I-B-G-Q-C-v5.tex b3c3286ec045fb348638d5f5e843561d66f17a21ed34c7ca5c189d3586953118

inputs/I-B-G-Q-C-v6.tex 28899a7966522ab65e100c7ab41de2a4235e6409e821a13a6514ebfc966e23c9

inputs/I-B-G-Q-C-v7.tex cf18a245ecb4e8d769aa755ef1c9b803fa1d64b4c4364516e20715cea61545f0

inputs/I-B-G-Q.api.tex 0d5a3149e1d56a5e0bf39c9ef727380e2864d8d6db1746c0541c730b2f24ec71

inputs/I-B-G.api.tex f692a066791ec479f7b3e9a938c7b1a43b944860791a171db48e4e6267d08e71

inputs/I-B-VP-2.api.tex 25de834d3d268a0d116a771441af5e4cf5582a822e71a700c9d0a9e44f7f5004

inputs/IX-E-3.api.tex 2bd298872d72300d58a64bdccf3d277cda1842533a7d8f226c2230306c9f7f15

inputs/IX-E.api.tex 184f8732fe2d674f4005cb617c67fa115af12b7760199c17ccbb193f55cf4fd4

inputs/Oai282-A.api.tex fc3ab1c2de3e01f6235c02459b22ee87462b13a72dee0059da00ff02d75a45ac

inputs/Oai282-B.api.tex d42f3cdeed913ef22ba08d5d088a760cd03b77a0a28d333195404c66b99a9063

inputs/VII-H.api.tex 30e35883a5241a93fa15538081abf8e2d96eacd43dd63e19056f1d0820d68c9c

../A-common/code/preamble_template.tex
2b3b5de7495a5e175a4545b6c4a01927c9b879d373a0a3b14a0e17947cd9d3b5

../A-common/code/build.py d688fe9663cece0624f4a648c695abf712b2f86add112fd61a3e3d2b4239d056

code/sheet.json 6fed8f3db31dc528f0de44714247007724e515332cba924e2fdebcc74062c89a

code/checks.py 16ef48e8c8a62a749a3b826fc15bb11c31892e7f35a61a6c6d8cf58939c43008

parts/01_front.tex d88cca22f89fecb175a095cb30f811be027705f8409df0438ee8636fe6d3e02b

parts/02_global.tex 5d708910d65d42b9b76142ce826d3502d7a2e26d239044d30929da20e15cf21e

parts/03_product.tex 49d0f3db91a36b6899bf4cef2381686f115f907ae3b83619207163 91f7eb03ab

parts/04_bridge.tex efe18c1a6b107d420d5c842a8159f276bbabc26e01251e53f162af7590ae35e7

parts/04b_state.tex a53e93d6a9ea3dcfbc5817d4497d3650a547148c0bc9197326e12c4316b1e79c

parts/04c_v3.tex 3b94c82b248cad82942c68a8ee4584cd5f16a151afccfe35861e62de3183dc37

parts/04d_v4.tex e72a3712ee5118081098f85eafb618632fa7d41628147217d72a0425b2385ce8

parts/04e_v5.tex 02f0d5b670a7ef207479aaee956196faea95eb24fa881eec03e36cb5b8b0ea31

parts/04f_v6.tex 39b9d4d7e20cc53a25e4b201ca244fd76f3a5ba4b419b32999230b0941421c1f

parts/04g_v7.tex 354bd83d00d246d772cd8c84273b9f1bd303471ffbb607b6723ccda7cfa968e3

parts/04h_v8.tex 1ff49a554ef922ee106c04b079bab1b7932980704e4000faa95493118399ac12

parts/05_release.tex 0c5e9babe99846aec27eb8365c34a66b6f50c1338fe624aee3fdb40abff76d50

parts/06_tables.tex 0524d2bb2f8bfd059d058a2ebc81e24baaeafe41ae2a15404d04745acf098082

parts/07_drafts.tex 67e35a3d0d13f62eebc166179b9791fc7faa3d3fbb37522ca0424d193c894af7

logs/checks.json e62d0d2e5ccfd4603a9d8c60774810482930dca545f5b3c72498faf59a7f4520

logs/inputs.sha256 828301408de7363c85bade8342cb080aaeac23b3e0055c86c970441ad7489470

../A-common/logs/site_fetch.txt 6cb47104d1f4d3d6041f8d547f88f6a202b4078d8f47bd8ab62e89b32705c813

midrule Frozen tree (before)

XVI-P ec965c15ae25cd3bb9cc2b3a134e966b19a1803c78d1fe5bf7554da2a8b3aaa6 OK

XVI-Q 0df9a0d10409d31ac0c2d0a47197e3f6e03c7ae258ec5fb775137f66cb0493c4 OK

XVI-R 57187bc6825f5c6ae88633b7279c6c0be57f7c4918fdbc525da4aaf958331edb OK

bottomrule endgroup The hashes of the delivered .tex and PDF are recorded in

MANIFEST-ClosePlans-A8.sha256, not here, because a file cannot contain its own hash.

References

[1] N. Cheung, *Black-Hole Information as an Access Problem: Enter, Quantum Recovery, and Complete Residuals*, figshare (2026). DOI: 10.6084/m9.figshare.34335024.v1. https://doi.org/10.6084/m9.figshare.34335024.v1.

[2] N. Cheung, *Operational Wormhole Certificates—Existence, Dimension-Dependent Scaling, Operational Functions, and Composition*, figshare (2026). DOI: 10.6084/m9.figshare.34549575.v1. https://doi.org/10.6084/m9.figshare.34549575.v1.

[3] N. Cheung, *Enter, Quantum Magic, and Quantum Gravity: Readout, Recovery, and the Limits of Geometric Response*, figshare (2026). DOI: 10.6084/m9.figshare.34335582.v1. https://doi.org/10.6084/m9.figshare.34335582.v1.

[4] N. Cheung, *Operational Consistency Does Not Select Lorentzian Geometry*, figshare (2026). DOI: 10.6084/m9.figshare.34441758.v1. https://doi.org/10.6084/m9.figshare.34441758.v1.

[5] N. Cheung, *Realization of Relational Time by Destructive Enter on a Clock Register*, figshare (2026). DOI: 10.6084/m9.figshare.34442907.v1. https://doi.org/10.6084/m9.figshare.34442907.v1.

[6] N. Cheung, ``The Enter Calculus: A Linear Quantum Lambda Calculus with an Operational Characterization of Measurement," *Frontiers in Physics* **14** (2026), article 1954943. DOI: 10.3389/fphy.2026.1954943. https://doi.org/10.3389/fphy.2026.1954943. Interface/operational background only; not a PDE premise.

[7] N. Cheung, *The Philosophy of Enter: Destructive Readout, Quantum Recovery, and the Refusal of Identification*, Cheung Ngo Medical Limited (2026). Framing essay; not a theorem premise. Programme note: https://enter.grok.me/enter.