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# I-B-G: Intrinsic and holonomy-twisted Green operators on a prescribed conical spacetime

*Self-adjoint extensions, diffraction, finite propagation, and supported transfer from flat-completion laboratories*

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**Abstract**

The regular conical product prescribed in I-B is not globally hyperbolic. Consequently, an intrinsic wave evolution requires an additional choice at the missing apex. We construct retarded and advanced inverses of the Klein–Gordon equation from a specified semibounded self-adjoint spatial extension, including flat complex line bundles with prescribed holonomy. We classify the rotation-diagonal, spatial-product extensions considered here and give their exact positivity criterion, distinguishing transverse bound states from the continuous bands they generate after adding the longitudinal direction.

For the Friedrichs extension we prove two-sided Green identities, an exact sequence on the extension-dependent spectral smooth domain, and finite propagation with respect to the intrinsic spatial distance. We derive a Sommerfeld–Carslaw–Dowker representation consisting of finitely many geometric terms and an apex-diffracted term. The latter begins at the broken-path length through the apex. Angular lift factors, local parallel trivialisations, image boundaries, and the advanced sign are specified explicitly.

A supported transfer theorem identifies the intrinsic scalar operator with the flat-completion operator before the first additional arrival. Its use in the Enter companion sheets also requires their independent retained-shadow and laboratory-margin hypotheses. A finite supplied-report catalogue is constructed without interpreting finite numerical agreement as a distributional identity or as acquisition of a continuum field. The reported numerical

discrepancies from the supplied draft are retained separately as unverified numerical data. General non-Friedrichs propagation, non-diagonal extension families, spinor and gauge constraints, uniform flat limits, states, and interacting products are not constructed.

### Bibliographic record


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## Book edition note (t427u)

Standing Enter refusals, scope vocabulary (proved here/supplied input/conditional/ not constructed/not claimed), the four-layer instrument box, and Scholar bibliography policy are frozen in the companion chapter *I-B-G Book Ledger* v1. O.1 remains NOT_CLOSED; XVI-N is not discharged; XV-B is not opened; no Host C, Page/island, or spacetime-emergence claim; no quantum coefficient is admitted; wired=false; SOURCE_DATE_EPOCH=1792627200. This sheet is a companion manuscript record and is *not* claimed to be Google Scholar-indexed. The published Enter Calculus article is cited only for finite-instrument / operational interface discussions, never as a PDE premise.

## Book expansion: separation table and front-coefficient note

toprule Landmark Relation Separation

midrule I-B Prescribed host / instrument Does not import Einstein dynamics

I-B-G-S Uses scalar kernels as input Spin constraints not a premise of scalar proofs

I-B-ECHO Uses front / SCD language Finite TV instruments not a premise here

I-B-G-Q series Free inverse language cited Source-test Ward ≠ interacting product

Big / Hawk / En09 Documentary ledger discipline Not premises of Green identities

OpenAI catalogue Coverage / invisible separations Documentary only

bottomrule

### Front coefficient versus Keller (book note)

For the untwisted scalar diffracted front, the high-frequency coefficient matches the *difference-angle* term of Keller's wedge coefficient for opening angle $\pi\alpha$, with the present Fourier convention; the cone has no reflecting-face term. Pure-image characters make the diffracted front vanish (compatible holonomy). The holonomy-twisted front is the bundle counterpart and is *not* claimed to already live inside Keller's untwisted wedge problem. Full asymptotic statements and residual-retention protocol for finite reports are developed in I-B-ECHO; this sheet supplies the continuum kernel identities they cite (supplied input there; proved here for the scalar SCD/Friedrichs core).

### The $E_0$ correction kernel (pointer)

Proposition already records the exact global distribution identity for the scale-invariant non-Friedrichs choice $E_0$ in one deficient mode. Book edition note: I-B-G-S cites this kernel when contrasting the idealized curvature-regular spin-2 branch with positive-core Mosco limits (I-B-G-S-B). The identity does *not* select interacting products or discharge any Q-series gate.

**Numerical protocol pointer**

Retained numerical samples in the appendix remain UNCERTIFIED. Certification artifacts are indexed in the I-B-G Book Ledger discrepancy appendix; none is promoted to a theorem. Residual-retention numerical discipline for finite instruments is the ECHO protocol (separation).

## Purpose, attribution, and logical scope

The spacetime companions X-A-ST, X-B-ST, X-C-ST, X-D-ST, V-B, IX-A-ST, IX-B-ST, IX-C-ST, and III-B use a supported developing presentation of the prescribed I-B host [15, 16, 17, 18, 19, 20, 21, 22, 23, 25]. Their global free scalar and Green operators belong to a separately declared Minkowski completion. A developing chart does not itself supply an intrinsic conical inverse.

This paper supplies such an inverse after an apex extension has been specified. It does not change the hypotheses or conclusions of those companions. The connection to their supported scalar calculations is a separate theorem (Theorem and Corollary ), not a reinterpretation of their original global analytical domains.

The analytical ingredients are classical: self-adjoint spectral theory; Bessel transforms; the Sommerfeld and Carslaw construction; Dowker's cone and flux calculations; and the geometric diffraction framework of Cheeger and Taylor [14, 12, 1, 2, 3, 4, 5, 6, 7]. Kay and Studer discuss the apex extension problem for cones and idealised cosmic strings [8]; the related radial inverse-square operators are treated explicitly by Dereziński and Richard [9]. These references provide mathematical context, not a derivation of the prescribed Enter geometry.

The Green-operator terminology is consistent with Bär–Ginoux–Pfäffle [10]. Their globally hyperbolic existence theorem is not invoked on the regular cone. Here existence comes from a declared spatial extension, and the exact sequence is proved on a declared spectral domain.

**What ``open'' means here.** An item marked **Open** is not resolved in this paper. This is not a claim that the corresponding problem is unknown in the literature. In particular, no assertion about the absence of results in an entire manuscript corpus is needed for any proof.

**Non-identifications.** The geometry, the apex condition, and any independent Aharonov–Bohm phase are inputs. The construction gives no Einstein dynamics, spacetime emergence, conical quantum state, Hadamard covariance, time-ordered product, or interacting algebra. The finite reporting interface introduced below does not measure an unknown continuum field.

## The prescribed host and its causal geometry

**Assumption (Prescribed I-B geometry).** Fix

$$0 < \mu < 2\pi, \qquad \alpha = 1 - \frac{\mu}{2\pi} \in (0,1), \qquad c > 0, \qquad m \geq 0.$$

The regular spatial cone and the regular spacetime are

$$C_\mu^\circ = (0,\infty) \times (\mathbb{R}/2\pi\mathbb{Z}), \qquad h = \mathrm{d}r^2 + \alpha^2 r^2 \mathrm{d}\varphi^2,$$

$$N = C_\mu^\circ \times \mathbb{R}_z, \qquad M = \mathbb{R}_T \times N, \qquad g = -c^2 \mathrm{d}T^2 + h + \mathrm{d}z^2.$$

The metric completion adds the apex worldsheet

$$\mathscr{W} = \mathbb{R}_T \times \{o\} \times \mathbb{R}_z,$$

but $\mathscr{W}$ is not part of $M$. The opening angle is $2\pi\alpha = 2\pi - \mu$. These are the prescribed background data of I-B [15].

Choose an origin $T_\Sigma$ and put

$$t = c(T - T_\Sigma), \qquad q = h + \mathrm{d}z^2, \qquad \mathrm{d}V = \alpha r \, \mathrm{d}r \, \mathrm{d}\varphi \, \mathrm{d}z.$$

Thus

$$g = -\mathrm{d}t^2 + q, \qquad \mathrm{dvol}_g = \mathrm{d}t \, \mathrm{d}V.$$

All propagation times and lengths below are measured in these $t$-units. A time bound in the $T$-coordinate is obtained by division by $c$.

## Signature and operator ledger

Let

$$\eta = \mathrm{d}t^2 - \mathrm{d}X^2 - \mathrm{d}Y^2 - \mathrm{d}z^2$$

on the analytical completion $\widehat{M} = \mathbb{R}^{1+3}$. On a supported developing region the map

$$\mathrm{dev}(t, r, \varphi, z) = (t, r\cos(\alpha\varphi), r\sin(\alpha\varphi), z)$$

satisfies

$$\mathrm{dev}^{*}\eta = -g.$$

Equivalently, if $j = \mathrm{dev}^{-1}$ is the chart embedding from the developed laboratory to the host, then $j^{*}g = -\eta$. This is the signature ledger used in X-A-ST and its successors [16, prop:signature-ledger, ass:host] [17, 18, 19, ass:host].

We use the scalar *equation operator*

$$P_m = -\mathrm{Box}_g + m^2 = \partial_t^2 - \Delta_q + m^2.$$

On the completion it is $\mathrm{Box}_\eta + m^2$. If a companion uses the action Hessian $-P_m$, its inverse is $-G_{\pm}$, not $G_{\pm}$. This distinction preserves the normalization of X-B-ST and XI-B [17, 28].

**Remark (Deficit convention).** Throughout, $\mu$ is the deficit, not the opening angle. In particular, $\alpha = \mu/(2\pi)$ would describe a different parametrisation. The Bessel orders in this paper are always $|n + \beta|/\alpha$ with $\alpha = 1 - \mu/(2\pi)$.

## Distance and the causal relation

For $x = (r, \varphi, z)$ and $x' = (r', \varphi', z')$, let

$$\psi \in (-\pi\alpha, \pi\alpha]$$

represent $\alpha(\varphi - \varphi')$ modulo $2\pi\alpha$, and write $\Delta z = z - z'$. Define

$$R_0(x, x')^2 = r^2 + r'\hat{}2 - 2rr'\cos\psi + \Delta z^2.$$

**Proposition (Intrinsic distance and causality).** *The intrinsic distance of N is*

$$d(x,x') = R_0(x,x').$$

*It is realised by a spatial geodesic lying entirely in N. Consequently,*

$$J_M^{\pm}(t',x') = \{(t,x) : \pm(t-t') \geq d(x,x')\}.$$

*For every compact $K \subset M$, $J_M^{\pm}(K)$ is closed in M.*

*Proof.* First consider the completed two-dimensional cone. It is a proper length space: bounded closed sets are compact in polar coordinates, with all angular coordinates identified at $r = 0$. A length-minimising curve therefore exists, by the usual compactness argument for equi-Lipschitz curves.

The developed straight segment with angular separation $\psi$ avoids the origin because $|\psi| \leq \pi\alpha < \pi$. It has length

$$(r^2 + r'^2 - 2rr'\cos\psi)^{1/2} < r + r'.$$

A minimiser cannot pass through the apex, since every such curve has length at least $r + r'$. Away from the apex a minimiser develops to a straight segment. Its continuous angular variation has absolute value less than $\pi$ and is congruent to $\psi$ modulo $2\pi\alpha$. The smallest possible absolute variation is $|\psi|$. Since the chord length is nondecreasing in the absolute angle on $[0,\pi]$, the displayed segment is minimising.

Taking the product with $\mathbb{R}_z$ gives : for every spatial curve, the integral triangle inequality bounds its length below by the Euclidean norm of its cone-length contribution and its total $z$-displacement. Equality is obtained by traversing the cone minimiser and the $z$-segment with constant relative speeds.

A future causal curve in the ultrastatic metric satisfies $|\dot{x}|_q \leq \dot{t}$, hence $t - t' \geq d(x,x')$. Conversely, a distance-realising spatial curve traversed at speed at most one gives a causal curve whenever this inequality holds. This proves .

Finally, if $p_j \in J_M^+(K)$ and $p_j \to p \in M$, choose $k_j \in K$ witnessing the relation. A subsequence has $k_j \to k \in K$;

continuity of $d$ preserves the inequality. The past case is identical.

**Remark (Local causal certificate versus global proof).** Inside a convex developing chart, a straight developed spatial segment of length $R_0$, together with elapsed time $R_0$, is null for the host metric. This is the local certificate recorded in I-B [15, lem:local-causal]. It does not by itself establish a global formula across a missing worldsheet. Proposition is a separate global argument for the specific cone with $0 < \alpha < 1$; its causal curves never pass through $\mathscr{W}$.

**Proposition (Failure of global hyperbolicity).** *The spacetime M is not globally hyperbolic.*

*Proof.* Fix $r_* > 0$, $\varphi_*$, $z_*$, and $a > r_*$. The points

$$p = (-a, r_*, \varphi_*, z_*), \qquad q = (a, r_*, \varphi_*, z_*)$$

have in their causal diamond the sequence

$$p_j = (0, r_j, \varphi_*, z_*), \qquad r_j \text{downarrow } 0.$$

Indeed, the required radial travel length is $r_* - r_j < a$. The sequence has no convergent subsequence in $M$. Thus this causal diamond is not compact.

This supplies directly the diagnosis used in I-B and the ST chart contracts [15, 21, 17].

## Angular lifts, geometric terms, and apex length

Choose lifts of $\varphi, \varphi'$ to $\mathbb{R}$, and set

$$\theta = \alpha(\varphi - \varphi').$$

Unlike $\psi$, $\theta$ is not reduced modulo $2\pi\alpha$. Put

$$\mathscr{K}(\theta) = \{k \in \mathbb{Z} : |\theta - 2\pi\alpha k| < \pi\},$$

$$R_k(\theta)^2 = r^2 + r'\hat{}2 - 2rr' \cos(\theta - 2\pi\alpha k) + \Delta z^2,$$

$$\rho(u)^2 = r^2 + r'\hat{}2 + 2rr' \cosh u + \Delta z^2, \qquad u \geq 0,$$

$$\rho_0 = \rho(0) = \sqrt{(r + r')^2 + \Delta z^2}.$$

The pair is called *generic* if

$$|\theta - 2\pi\alpha k| \neq \pi \quad \text{for every} k \in \mathbb{Z}.$$

These excluded loci are the geometric image boundaries.

If $\theta = \psi$, then $0 \in \mathcal{K}(\psi)$, and

$$d(x, x') = R_0 \leq R_k < \rho_0 \qquad (k \in \mathcal{K}(\psi)).$$

At the angular cut $|\psi| = \pi\alpha$, two shortest geometric terms can have equal length. Thus it is not correct globally to assert $R_k > R_0$ for every $k \neq 0$. The strict inequality does hold for these additional terms in an angularly small chart defined in Section .

### Orientation of geometric holonomy

With the displayed developing map,

$$\mathrm{dev}(r, \varphi + 2\pi) = \mathrm{Rot}\,(2\pi\alpha)\,\mathrm{dev}(r, \varphi) = \mathrm{Rot}\,(-\mu)\,\mathrm{dev}(r, \varphi).$$

The inverse gluing generator is therefore

$$J_\mu(t, X, Y, z) = (t, \mathrm{Rot}\,(\mu)(X, Y), z).$$

Calling $J_\mu$ the positively oriented holonomy, as in the I-B convention, amounts to choosing the corresponding gluing or inverse-meridian orientation [15, prop:holonomy]. Both generators fix $t, z$; they have no translational part. Reversing the chosen generator reverses every holonomy sign below. The distinction is recorded rather than hidden in the notation.

## Flat bundles and the restricted extension ledger

### Twisted sections and deficiency indices

For $\beta \in [0,1)$, represent a section of a flat complex line bundle by a function on the angular cover satisfying

$$u(r, \varphi + 2\pi, z) = e^{2\pi i \beta} u(r, \varphi, z).$$

Its norm is taken over one angular period. Denote the resulting Hilbert space by $\mathscr{H}_\beta = L^2_\beta(N, \mathrm{d}V)$. In this lifted parallel representation the flat covariant derivative is the ordinary derivative.

Let $A_{0,\beta} = -\Delta_q$ on compactly supported smooth sections. Angular separation gives

$$u = \sum_{n \in \mathbb{Z}} \frac{e^{\mathrm{i}(n+\beta)\varphi}}{\sqrt{2\pi\alpha}} u_n(r, z), \qquad \nu_n = \frac{|n+\beta|}{\alpha}.$$

The radial expression on $L^2((0,\infty), r\,\mathrm{d}r)$ is

$$\ell_\nu = -\partial_r^2 - \frac{1}{r}\partial_r + \frac{\nu^2}{r^2}.$$

**Lemma (Deficiency count).** *The expression $\ell_\nu$ is limit circle at zero exactly when $0 \le \nu < 1$, and is limit point at infinity. The transverse minimal operator has deficiency indices $(d_\beta, d_\beta)$, where*

$$d_\beta = \#\{n \in \mathbb{Z} : |n+\beta| < \alpha\}.$$

*In particular:*

***1.*** *$d_0 = 1$, with deficient mode $n = 0$.*

***2.*** *For $0 < \beta < 1$, only $n = 0, -1$ can be deficient. They are deficient respectively when $\beta < \alpha$ and $1 - \beta < \alpha$.*

***3.*** *Two deficient modes occur precisely when $1 - \alpha < \beta < \alpha$, which requires $\alpha > 1/2$. The full minimal operator on $N$ has infinite deficiency indices if $d_\beta > 0$, and is essentially self-adjoint if $d_\beta = 0$.*

*Proof.* At zero the solutions at spectral parameter zero are $r^\nu, r^{-\nu}$ for $\nu > 0$, and $1, \log r$ for $\nu = 0$. Both are square integrable with weight $r$ exactly for $\nu < 1$. At infinity the expression is limit point, for example after conjugating by $f \mapsto r^{1/2} f$ to the half-line Schrödinger expression

$$-\partial_r^2 + (\nu^2 - \tfrac{1}{4})r^{-2}.$$

Weyl's alternative gives the radial assertion [14, 9]. Counting the angular modes gives the transverse indices.

Fourier transformation in $z$ gives the radial expressions shifted by $k_z^2$. When a deficient transverse mode exists, its boundary data may vary over an infinite-dimensional space of longitudinal functions. If no such mode exists, the direct integral of the unique self-adjoint fibre closures is the closure of the full operator.

**Assumption (Product and rotation-diagonal restriction).** We consider only extensions of the form

$$B = B_\perp \otimes 1 + 1 \otimes (-\partial_z^2) + m^2,$$

where $B_\perp$ is a self-adjoint extension of the transverse minimal operator and commutes with rotations. The transverse boundary parameters are independent of $k_z$.

This restriction is stronger than rotation and $z$-translation invariance alone. Such invariance can also permit measurable $k_z$-dependent boundary conditions. Those extensions, longitudinally nonlocal conditions, and non-product extensions are not classified here.

The rotation characters $e^{\mathrm{i}(n+\beta)\gamma}$ are distinct. Thus $B_\perp$ is diagonal in $n$. Fixing a reference length $\ell_0 > 0$, the deficient radial domains have boundary coefficients of the form

$$\begin{aligned} \mathrm{ll}\nu = 0: &\quad f(r) = a\log(r/\ell_0) + b + o(1), \\ 0 < \nu < 1: &\quad f(r) = ar^{-\nu} + br^{\nu} + o(r^{\nu}). \end{aligned}$$

The coefficients can equivalently be defined by limiting Wronskians. The corresponding Green boundary form shows that the self-adjoint one-mode conditions are:

• $\nu = 0$: the Friedrichs condition $a = 0$, or

$$\mathrm{L}_L: \qquad f(r) = a\log(r/L) + o(1), \qquad 0 < L < \infty;$$

• $0 < \nu < 1$: the Friedrichs condition $a = 0$, or

$$\mathrm{E}_\lambda: \qquad b = \lambda a, \qquad \lambda \in \mathbb{R}.$$

We also write $\lambda = \infty$ for Friedrichs. Indeed, for $0 < \nu < 1$ the boundary form is a nonzero real multiple of $\overline{a}_f b_g - \overline{b}_f a_g$, so its one-dimensional self-adjoint subspaces have exactly this real projective parametrisation. The logarithmic case is analogous.

## Positivity and the longitudinal spectral bands

**Theorem (Corrected positivity ledger).** *Under Assumption , the following hold.*

***1.*** *The Friedrichs extension $A_{F,\beta}$ of $-\Delta_q$ is nonnegative. Its form domain is the closure of compactly supported smooth sections in*

$$\|u\|^2 + \|\nabla_\beta u\|^2.$$

***2.*** *In the logarithmic mode, $\mathrm{L}_L$ has exactly one transverse negative eigenvalue*

$$-\kappa_L^2, \qquad \kappa_L = \frac{2e^{-\gamma_E}}{L},$$

*with radial eigenfunction $K_0(\kappa_L r)$.*

***3.*** *In a mode $0 < \nu < 1$, $\mathrm{E}_\lambda$ has a transverse negative eigenvalue exactly when $\lambda < 0$. It is unique in that mode and is $-\kappa^2$, where*

$$\lambda = \frac{\Gamma(-\nu)}{\Gamma(\nu)}\left(\frac{\kappa}{2}\right)^{2\nu}.$$

*For $\lambda \in [0, \infty]$ the transverse extension is nonnegative.*

***4.*** *Every transverse ledger extension has essential spectrum $[0, \infty)$, with only the finitely many negative eigenvalues just described below it. Let*

$$\kappa_* = \max\left(\{0\} \cup \{\kappa_j : \ -\kappa_j^2 \text{is a transverse negative eigenvalue}\}\right).$$

*Then the full product operator satisfies*

$$\operatorname{spec}(B) = \operatorname{spec}_{\mathrm{ess}}(B) = [m^2 - \kappa_*^2, \infty), \qquad B \geq 0 \ \Leftrightarrow m \geq \kappa_*.$$

*The full operator has no $L^2(N)$ point spectrum.*

*Proof.* The first assertion is the Friedrichs construction applied to the nonnegative covariant Dirichlet form [14].

At a negative radial energy $-\kappa^2$, square integrability at infinity forces the solution to be a multiple of $K_\nu(\kappa r)$. For $\nu = 0$,

$$K_0(\kappa r) = -\log(\kappa r/2) - \gamma_E + o(1) = -\log(r/L) + o(1)$$

if and only if $L = 2e^{-\gamma_E}/\kappa$. This gives one and only one negative eigenvalue for each finite $L$.

For $0 < \nu < 1$,

$$K_\nu(\kappa r) = \frac{\Gamma(\nu)}{2}\left(\frac{\kappa r}{2}\right)^{-\nu} + \frac{\Gamma(-\nu)}{2}\left(\frac{\kappa r}{2}\right)^{\nu} + o(r^\nu).$$

The ratio of the two boundary coefficients is . Since $\Gamma(-\nu) < 0$, its right-hand side is a bijection from $\kappa > 0$ onto $\lambda < 0$.

The Friedrichs radial spectrum is $[0, \infty)$, as follows from the Bessel spectral transform. Changing finitely many endpoint conditions gives a finite-rank transverse resolvent difference, so it does not change the transverse essential spectrum. The preceding square-integrable solutions account for all negative eigenvalues. There is no transverse zero eigenfunction: the solution permitted at zero cannot also be square integrable at infinity. These facts also follow from the explicit half-line analysis in [9].

Fourier transformation in $z$ turns $B$ into the direct integral of $B_\perp + k_z^2 + m^2$. Each transverse bound state therefore generates a continuous band $[m^2 - \kappa_j^2, \infty)$, not a full-space bound eigenfunction. The transverse continuum generates $[m^2, \infty)$. Their union is . For any fixed spectral value, the equation $k_z^2 + \lambda + m^2 = E$ holds on a Lebesgue-null set of $k_z$ for each transverse spectral value $\lambda$; the direct-integral spectral representation therefore has no full-space eigenvectors.

**Remark (What the mass can and cannot cure).** For $m = 0, \beta = 0$, Friedrichs is the only nonnegative extension in this ledger. For $m > 0$, a logarithmic extension is nonnegative precisely when $m \geq \kappa_L$. The same mass-shift

qualification applies to every $\lambda < 0$ power-law mode. It would be incorrect to label all such massive operators ``not nonnegative" merely because their transverse extension has a negative eigenvalue.

The geometry does not select $L$, $\lambda$, or the Friedrichs choice. The latter is the finite-energy closure of compactly supported data. Kay–Studer's discussion [8] supplies physical context, not an additional premise selecting an extension.

## Green operators on an extension-dependent spectral domain

Let $B$ be any extension in the restricted ledger. By Theorem , $B$ is bounded below. Choose $b > 0$ so that $B + b \geq 1$, and define

$$\mathscr{H}_B^\infty = \bigcap_{k=0}^{\infty} \operatorname{dom}(B^k), \qquad p_k(v) = \|(B+b)^k v\|.$$

This is a Fréchet space; its topology is independent of the admissible shift $b$. Put

$$\mathcal{D}_{\mathrm{tc}_B} = C_c^\infty(\mathbb{R}; \mathscr{H}_B^\infty), \qquad \mathcal{E}_B = C^\infty(\mathbb{R}; \mathscr{H}_B^\infty).$$

The first space has its usual inductive-limit topology over compact time supports; it is not asserted to be a Fréchet space.

For real $\lambda$ set

$$w_t(\lambda) = \begin{cases} \sin(t\sqrt{\lambda})/\sqrt{\lambda}, & \lambda > 0, \\ t, & \lambda = 0, \\ \sinh(t\sqrt{-\lambda})/\sqrt{-\lambda}, & \lambda < 0, \end{cases} \qquad W_B(t) = w_t(B).$$

The spectral multiplier is bounded for each fixed $t$ because $B$ is bounded below. Functional calculus, rather than convergence of an operator power series on arbitrary smooth vectors, is the definition.

**Definition (Retarded, advanced, and causal operators).** For $f \in \mathcal{D}_{\mathrm{tc}_B}$, define

$$(G_+^B f)(t) = \int_{-\infty}^{t} W_B(t-s) f(s)\, \mathrm{d}s,$$

$$(G_-^B f)(t) = -\int_{t}^{\infty} W_B(t-s) f(s)\, \mathrm{d}s = \int_{t}^{\infty} W_B(s-t) f(s)\, \mathrm{d}s,$$

$$E_B = G_+^B - G_-^B.$$

Thus, in time-convolution notation,

$$G_\pm^B(t) = \pm\theta(\pm t)W_B(t).$$

The minus sign in the first advanced integral is essential.

**Theorem (Green identities and spectral exactness).** *For every restricted ledger extension B:*

***1.*** *Compactly supported smooth spacetime sections belong continuously to* $\mathcal{D}_{\mathrm{tc}_B}$*, and*

$$G_\pm^B : \mathcal{D}_{\mathrm{tc}_B} \to \mathcal{E}_B$$

*are continuous.*

***2.*** *With* $P_B = \partial_t^2 + B$*,*

$$P_B G_\pm^B f = G_\pm^B P_B f = f \qquad (f \in \mathcal{D}_{\mathrm{tc}_B}).$$

*On compactly supported smooth sections,* $P_B = P_m$*.*

***3.*** $G_+^B f$ *vanishes before the time support of f, and* $G_-^B f$ *vanishes after it.*

***4.*** *The sequence*

$$0 \to \mathcal{D}_{\mathrm{tc}_B} \mathrm{xrightarrow} P_B \mathcal{D}_{\mathrm{tc}_B} \mathrm{xrightarrow} E_B \mathcal{E}_B \mathrm{xrightarrow} P_B \mathcal{E}_B \to 0$$

*is exact.*

***5.*** *In the Hermitian spacetime pairing,*

$$\langle g, G_+^B f\rangle = \langle G_-^B g, f\rangle, \qquad \langle g, E_B f\rangle = -\langle E_B g, f\rangle \qquad (f, g \in \mathcal{D}_{\mathrm{tc}_B}).$$

*These are formal adjoint identities on the stated test domain, not assertions of self-adjointness or skew-adjointness of unbounded spacetime operators.*

***6.*** $G_\pm^B f$ *is smooth on the regular spacetime M.*

*Proof.* The spatial differential expression maps compactly supported smooth sections to themselves. Every such section therefore lies in every power domain of $B$. The same statement with compact-support seminorm estimates proves the asserted continuous inclusion.

On compact time intervals the multipliers $w_t$ and $\partial_t w_t$ are bounded uniformly on the spectrum; higher time derivatives are controlled by finitely many additional graph seminorms. Differentiation under the integral in consequently gives

$$W_B(0) = 0, \qquad \dot{W}_B(0) = \mathrm{id}, \qquad \ddot{W}_B(t) = -BW_B(t)$$

on $\mathscr{H}_B^\infty$. Duhamel differentiation proves the right inverse identities. Integration by parts in $s$, using compact time support, proves the left inverse identities. The support-in-time assertions are immediate.

For completeness, the homogeneous Cauchy problem is unique in $\mathcal{E}_B$: applying bounded spectral projections of $B$ reduces it to a bounded-operator Cauchy problem; the projections tend strongly to the identity. Its solution with data $u_0, u_1 \in \mathscr{H}_B^\infty$ is

$$u(t) = \dot{W}_B(t)u_0 + W_B(t)u_1.$$

If $P_B u = 0$ and $u$ has compact time support, then $u = G_+^B P_B u = 0$, so the first arrow in is injective. The inverse identities give $E_B P_B = 0$ and $P_B E_B = 0$. If $E_B f = 0$, then $u = G_+^B f = G_-^B f$ has time support in the compact interval spanning $\mathrm{supp}_t f$, and $P_B u = f$. This proves exactness at the second space.

If $P_B u = 0$, choose $\chi \in C^\infty(\mathbb{R})$ equal to zero for $t < 0$ and one for $t > 1$, and set

$$f = P_B(\chi u) = [P_B, \chi]u \in \mathcal{D}_{\mathrm{tc}_B}.$$

Cauchy uniqueness gives

$$G_+^B f = \chi u, \qquad G_-^B f = -(1-\chi)u,$$

hence $E_B f = u$. Finally, for arbitrary $v \in \mathcal{E}_B$,

$$u(t) = \int_0^t W_B(t-s)v(s)\,\mathrm{d}s$$

is a smooth spectral-domain solution of $P_B u = v$. This proves the last surjectivity.

Since $W_B(t)^* = W_B(t)$ and $W_B(-t) = -W_B(t)$, Fubini's theorem on the compact time supports of the paired tests gives the adjoint identities.

Interior elliptic regularity gives continuous local embeddings

$$\mathrm{dom}\,(B^k) \mathrm{hookrightarrow} H^{2k}_{\mathrm{loc}}(N).$$

Together with the spectral time regularity, these imply smoothness on $M$.

**Remark (The unresolved compact-spacetime sequence).** The exact sequence depends on $B$. It is not an exactness theorem on the usual compact-spacetime test space. On that space we have proved only

$$P_m C_c^\infty(M) \subseteq \ker E_B \cap C_c^\infty(M).$$

The reverse inclusion is not established here. A time-compact spectral solution need not have support bounded away from the removed worldsheet. This distinction must not be erased when comparing with Minkowski equation quotients or free-field algebras.

## Finite propagation for the Friedrichs extension

**Theorem (Intrinsic finite propagation).** *Let $B = A_{F,\beta} + m^2$. For every compactly supported smooth spacetime section $f$,*

$$\mathrm{supp}\, G^B_\pm f \subseteq J^\pm_M(\mathrm{supp}\, f).$$

*The causal relation is the intrinsic relation , not an auxiliary completion relation.*

*Proof.* We prove the domain-of-dependence statement for the homogeneous equation and then use Duhamel's formula.

Let $u(0) = u_0, \dot{u}(0) = u_1$, with smooth spectral data vanishing on $B_d(x_0, \rho)$. For $t \geq 0$, choose a smooth nonincreasing $\varphi_\varepsilon$, equal to one on $(-\infty, -\varepsilon]$ and zero on $[0, \infty)$, and put

$$\chi_t(x) = \varphi_\varepsilon(d(x, x_0) + t - \rho).$$

The distance is Lipschitz and $|\nabla d| \leq 1$ almost everywhere.

Multiplication by bounded Lipschitz functions with bounded gradient preserves the Friedrichs form domain $Q_F$. Indeed, approximate a form-domain section by compactly supported smooth sections, apply the Leibniz estimate, and smooth the multiplier on the compact regular supports. Thus all integrations by parts below are identities in the closed form, including when the support of $\chi_t$ approaches the apex.

Use the augmented local energy

$$e = |\dot{u}|^2 + |\nabla_\beta u|^2 + (m^2 + 1)|u|^2, \qquad \mathscr{E}(t) = \int_N \chi_t e \, \mathrm{d}V.$$

The extra $|u|^2$ term also handles the massless case. The form identity for $B$ yields

$$\mathscr{E}'(t) = \int (\partial_t \chi_t) e \, \mathrm{d}V - 2\mathrm{Re} \int \bar{\dot{u}} \nabla \chi_t \cdot \nabla_\beta u \, \mathrm{d}V$$

$$+2\mathrm{Re} \int \chi_t \bar{\dot{u}} \dot{u} \, \mathrm{d}V.$$

Since $\varphi_\varepsilon' \leq 0$,

$$\int (\partial_t \chi_t) e - 2\mathrm{Re} \int \bar{\dot{u}} \nabla \chi_t \cdot \nabla_\beta u \leq \int \varphi_\varepsilon' (e - 2|\dot{u}||\nabla_\beta u|) \leq 0.$$

Also $2|\dot{u}u| \leq |u|^2 + |\dot{u}|^2 \leq e$. Hence

$$\mathscr{E}'(t) \leq \mathscr{E}(t), \qquad \mathscr{E}(0) = 0.$$

Gronwall's inequality gives $\mathscr{E}(t) = 0$. Letting $\varepsilon$downarrow$0$ proves that $u(t)$ vanishes on $B_d(x_0, \rho - t)$. Time reversal gives the negative-time statement.

It follows that $W_B(t)$ propagates spatial support by at most $|t|$. Duhamel's formula then gives ; closedness of

$J_M^{\pm}(\operatorname{supp} f)$ was proved in Proposition .

**Remark (Why no apex flux was discarded).** The argument uses the closed Friedrichs form rather than an integration over $r > \varepsilon$ followed by an unjustified boundary limit. It does not prove finite propagation for the general $\mathrm{L}_L$ or $\mathrm{E}_\lambda$ domains. The special $\mathrm{E}_0$ case is treated separately below. The result is consistent with the general conic diffraction theory of Cheeger–Taylor [6, 7].

## The Sommerfeld–Carslaw–Dowker kernel

In this section $B = A_{F,\beta} + m^2$, and all spatial kernels are relative to $\mathrm{d}V$.

### The diffraction weight

For $u > 0$, define

$$\Phi_\beta(u,\theta) = \sum_{n\in\mathbb{Z}} \sin(\pi\nu_n) e^{-\nu_n u} e^{\mathrm{i}(n+\beta)\theta/\alpha}.$$

This series converges absolutely.

**Lemma (Closed weight and its boundary control).** *Set*

$$a = \frac{\pi}{\alpha}, \qquad b = \frac{u}{\alpha}, \qquad \xi = \frac{\theta}{\alpha},$$

*and*

$$F(\gamma, w) = \frac{1}{2\mathrm{i}}\left(\frac{e^{\gamma(w+\mathrm{i}a)}}{1-e^{w+\mathrm{i}a}} - \frac{e^{\gamma(w-\mathrm{i}a)}}{1-e^{w-\mathrm{i}a}}\right).$$

*Then*

$$\Phi_\beta(u,\theta) = F(\beta, -b+\mathrm{i}\xi) + F(1-\beta, -b-\mathrm{i}\xi).$$

*At a generic angle this extends continuously to $u = 0$. For $\beta = 0$,*

$$\Phi_0(u,\theta) = \frac{1}{2} D_\alpha(u,\theta), \qquad D_\alpha = \sum_{\pm} \frac{\sin((\pi\pm\theta)/\alpha)}{\cosh(u/\alpha) - \cos((\pi\pm\theta)/\alpha)}.$$

*The function $\Phi_\beta$ is jointly locally integrable in $(u,\theta)$ up to $u = 0$, including image boundaries, and decays*

*exponentially as* $u \to \infty$.

*Proof.* Split the series into $n \geq 0$ and $n \leq -1$, and write $\sin x = (e^{ix} - e^{-ix})/(2i)$. The resulting two geometric sums give ; the $n = 0, \beta = 0$ contribution is zero. Combining conjugate geometric denominators gives .

The possible boundary poles are precisely $\theta \pm \pi \in 2\pi\alpha\mathbb{Z}$. Near each such angle $\theta_j$, the closed form is bounded by a constant times

$$1 + (u^2 + (\theta - \theta_j)^2)^{-1/2}.$$

This is locally integrable in the two variables. Away from those angles, all denominators remain nonzero at $u = 0$.

For $0 < \beta < 1$, exponential decay follows with rate at least $\min(\beta, 1 - \beta)/\alpha$. For $\beta = 0$, the zero mode does not contribute, and a rate $1/\alpha$ suffices.

The untwisted expression is the Carslaw weight; the twisted version is the corresponding cone-flux construction [2, 3, 4, 5].

## A precise class of kernel identities

We do not assert a master formula for arbitrary bounded Borel functions. The following theorem concerns the heat semigroup, the positive resolvent, and the retarded wave kernel, with the latter interpreted distributionally.

For any radial flat kernel $F(R)$ for which the following expression is defined, write

$$\mathcal{S}_\beta[F] = \sum_{k \in \mathcal{K}(\theta)} e^{2\pi i \beta k} F(R_k) - \frac{1}{\pi\alpha} \int_0^\infty \Phi_\beta(u, \theta) F(\rho(u))\, du.$$

**Theorem (Heat, resolvent, and wave representation).** *The following identities hold.*

***1.*** *The massless spatial heat kernel is*

$$H_\beta(\tau; x, x') = \frac{e^{-\Delta z^2/(4\tau)}}{\sqrt{4\pi\tau}} \frac{e^{-(r^2 + r'^2)/(4\tau)}}{4\pi\alpha\tau}$$

$$\times \sum_{n \in \mathbb{Z}} e^{i(n + \beta)(\varphi - \varphi')} I_{\nu_n}\left(\frac{rr'}{2\tau}\right), \qquad \tau > 0.$$

*At generic pairs, the heat kernel of B equals*

$(\partial_t^2 - \Delta_{\mathbb{R}^3} + m^2).\ For (R > 0), F_m^+(t,R) = \frac{\delta(t-R)}{4\pi R} - \theta(t-R)\frac{mJ_1(m\sqrt{t^2-R^2})}{4\pi\sqrt{t^2-R^2}}, where the second term is zero at (m = 0), and its quotient has its continuous value at the front. The intrinsic retarded kernel is G_+(t;x,x') =$

*The last term means its locally integrable one-sided extension at* $t = \rho_0$.

***2.*** *The formulas are globally bundle-valued distribution identities with the joint-integration prescription in the proof. No arbitrary pointwise value at an image boundary is required. The advanced kernel is*

$$G_-(t;x,x') = G_+(-t;x,x').$$

*Proof. Step 1: the mode heat kernel.* The Friedrichs radial operator is diagonalised by the Bessel transform with kernel $J_\nu(kr)$. Weber's exponential integral gives

$$\int_0^\infty e^{-\tau k^2} J_\nu(kr) J_\nu(kr')\, k\, \mathrm{d}k = \frac{1}{2\tau} e^{-(r^2 + r'^2)/(4\tau)} I_\nu\left(\frac{rr'}{2\tau}\right).$$

Together with the angular normalization and the one-dimensional longitudinal heat kernel, this proves . We use the classical integral in [12]; equivalent Bessel identities are recorded in [13].

*Step 2: the angular summation.* For $\zeta > 0$, set

$$S(\zeta,\theta) = \sum_n I_{\nu_n}(\zeta) e^{\mathrm{i}(n+\beta)\theta/\alpha}.$$

Use the integral representation

$$I_\nu(\zeta) = \frac{1}{\pi}\int_0^\pi e^{\zeta\cos v}\cos(\nu v)\,\mathrm{d}v - \frac{\sin(\pi\nu)}{\pi}\int_0^\infty e^{-\zeta\cosh u - \nu u}\,\mathrm{d}u,$$

valid for the present parameters [12, 13]. The first integral is the *n*-th twisted Fourier coefficient of the piecewise smooth function

$$h_\zeta(\theta) = \alpha \sum_{k\in\mathbb{Z}} e^{2\pi i\beta k} e^{\zeta\cos(\theta - 2\pi\alpha k)} 1_{\{|\theta - 2\pi\alpha k| < \pi\}}.$$

Indeed, integrating over one angular period and changing variables in each summand gives the coefficient

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\zeta\cos v} e^{-\mathrm{i}(n+\beta)v/\alpha}\,\mathrm{d}v.$$

Its imaginary part vanishes, and its real part is the first Bessel integral.

Insert the Abel factor $e^{-\varepsilon v_n}$. The first Fourier series tends to $h_\zeta$ at every generic angle and in local angular $L^1$. The second becomes

$$-\frac{1}{\pi}\int_0^{\infty} e^{-\zeta\cosh u}\Phi_\beta(u+\varepsilon,\theta)\,\mathrm{d}u.$$

Lemma supplies local joint integrability, uniformly for small $\varepsilon$; at infinity the factor $e^{-\zeta\cosh u}$ dominates. Hence

$$S(\zeta,\theta) = \alpha \sum_{k\in\mathcal{K}(\theta)} e^{2\pi\mathrm{i}\beta k} e^{\zeta\cos(\theta-2\pi\alpha k)} - \frac{1}{\pi}\int_0^{\infty} e^{-\zeta\cosh u}\Phi_\beta(u,\theta)\,\mathrm{d}u.$$

Multiplying by the remaining factors in , with $\zeta = rr'/(2\tau)$, proves the heat identity.

*Step 3: the resolvent.* Since $B \geq 0$,

$$(B+s^2)^{-1} = \int_0^{\infty} e^{-\tau s^2} e^{-\tau B}\,\mathrm{d}\tau.$$

The flat heat integral is $e^{-\sqrt{s^2+m^2}R}/(4\pi R)$. For generic pairs the diffraction weight is bounded at $u = 0$, and the exponential in $\rho(u)$ controls infinity. After compact spatial smearing, the joint bound in Lemma justifies the same interchanges across image boundaries. This proves the resolvent formula.

*Step 4: the global wave distribution.* Define the right-hand side of first by joint testing in time and the spatial pair. The geometric terms are the usual flat wave distributions on the indicated angular sectors, with their locally integrable sector indicators.

For the diffracted term, compact time support bounds $u$: $F_m^+(t,\rho(u))$ vanishes unless $\rho(u)$ is no larger than the largest positive time in the test support. On compact regular spatial sets, $\rho(u) \geq r + r' > 0$. After time testing,

$$\left\langle F_m^+(t,\rho), a(t)\right\rangle$$

is a bounded smooth function of $\rho > 0$ on the relevant compact range. Lemma therefore makes the joint $u$-angular

integration locally integrable, including at $u = 0$ and an image boundary. This gives an unambiguous global distribution.

Its Laplace transform is the resolvent formula from Step 3, because

$$\int_0^\infty e^{-st} F_m^+(t,R)\,\mathrm{d}t = \frac{e^{-\sqrt{s^2+m^2}R}}{4\pi R}.$$

On the spectral side,

$$\int_0^\infty e^{-st} W_B(t)\,\mathrm{d}t = (B+s^2)^{-1}.$$

Both sides are causal distributions of at most polynomial growth after compact spatial smearing. Injectivity of the Laplace transform therefore proves globally.

For $m = 0$, integrating the delta in $u$ uses

$$\rho'(u) = \frac{rr'\sinh u}{\rho(u)}.$$

This gives . At a generic pair, $\Phi_\beta$ is bounded at zero and the resulting edge is of order $(t-\rho_0)^{-1/2}$. Finally, the advanced identity follows from Definition and oddness of $W_B$.

**Remark (Why a null-set argument would not suffice).** The extension across image boundaries in the proof is a joint-distribution construction. It is not justified merely by saying that the exceptional angular pairs have measure zero: a distribution can be supported on such a set. The heat and resolvent identities, together with Laplace uniqueness, fix the global distribution.

### Bundle covariance and local gauges

The kernel is a section of $\mathrm{Hom}\,(L_{\beta,x'}, L_{\beta,x})$. For example,

$$G_+(t;r,\varphi+2\pi,z;x') = e^{2\pi i\beta} G_+(t;r,\varphi,z;x').$$

If

$$\theta = \psi + 2\pi\alpha\ell, \qquad \ell \in \mathbb{Z},$$

then every formula reduced to $\psi$ acquires the overall factor

$$G_+(t;\theta) = e^{2\pi i\beta\ell} G_+(t;\psi).$$

The same identity holds for $\Phi_\beta$. Omitting this factor would destroy bundle covariance.

On a simply connected chart, the lifted representation is a *parallel trivialisation*; the differential operator is the ordinary scalar Klein–Gordon operator. By contrast, the periodic gauge

$$v = e^{-i\beta\varphi} u$$

has covariant derivative $\partial_\varphi + i\beta$, and its kernel is

$$K_{\mathrm{per}}(x,x') = e^{-i\beta\varphi} K_{\mathrm{par}}(x,x') e^{i\beta\varphi'}.$$

In that gauge the direct term has the corresponding endpoint phase. It is not the ordinary flat kernel with unit direct weight. All flat comparisons below use a local parallel trivialisation.

**Kernel structure and qualifications**

**Corollary (Geometric and diffracted contributions).** *For the Friedrichs extension:*

***1.*** *In a lift with $\theta = \psi$, the $k = 0$ term is the direct flat term at $d(x,x') = R_0$. Every additional geometric term arrives at $R_k \geq R_0$.*

***2.*** *The diffracted contribution is supported in $t \geq \rho_0$. At a generic pair its massless front coefficient is*

$$\lim_{t\,\mathrm{downarrow}\,\rho_0} \sqrt{t^2 - \rho_0^2}\, G_+^{\mathrm{dif}}(t;x,x') = -\frac{\Phi_\beta(0,\theta)}{4\pi^2\alpha\sqrt{rr'}}.$$

*This coefficient may vanish for a particular pair. A nonzero diffraction weight gives a late-time contribution, but not necessarily a nonzero value at every angle and every later time.*

***3.*** *The diffraction weight vanishes identically precisely when*

$$\frac{1}{\alpha} = N \in \mathbb{N}, \qquad N\beta \in \mathbb{Z}.$$

*In this case the kernel is the finite character-weighted N-image kernel of the corresponding cyclic quotient. For* $\beta = 0, \alpha = 1/2$,

$$G_{+}(t; x, x') = F_{m}^{+}(t, R_0) + F_{m}^{+}(t, \overline{R}),$$

*where*

$$\overline{R}^2 = r^2 + r'^2 + 2rr'\cos\psi + \Delta z^2.$$

***4.*** *Every term is supported in the intrinsic causal future. At an angular cut, two geometric fronts can lie on its boundary. Away from that cut, only the shortest geometric front does so.*

*Proof.* The support assertions follow from and . Near $t = \rho_0$,

$$\sinh u_0 \sim \frac{\sqrt{t^2 - \rho_0^2}}{\sqrt{rr'}},$$

which proves .

If holds, every $\sin(\pi\nu_n)$ vanishes. Conversely, if $\Phi_\beta$ vanishes identically, its absolutely convergent angular Fourier series at each $u > 0$ has zero coefficients, so every $\nu_n$ is an integer. The successive positive orders differ by $1/\alpha$; therefore $1/\alpha$ is an integer, and $\beta/\alpha = N\beta$ is an integer. For a generic angle the geometric interval contains exactly $N$ indices. The remaining statements follow from the length comparison and Proposition .

For $m = 0$, the flat kernel satisfies Huygens' principle, whereas nonzero diffraction produces a wake beyond the geometric fronts. This does not mean propagation outside the intrinsic causal relation. With nontrivial flux, diffraction can persist even at an untwisted pure-image cone; it disappears there only for the compatible characters in [11, 5].

**The scale-invariant non-Friedrichs choice**

**Proposition (The extension $E_0$).** *Use* $E_0$ *in one deficient mode n, with* $0 < \nu = \nu_n < 1$, *and Friedrichs in the other*

*modes. Then* $B \geq 0$*, and*

$$G_+^B - G_+^F = \frac{2\sin(\pi\nu)}{\pi\alpha} e^{\mathrm{i}(n+\beta)(\varphi-\varphi')} \times \int_0^\infty \cosh(\nu u) F_m^+(t, \rho(u))\, \mathrm{d}u.$$

*This is a global distribution identity. The correction is supported in* $t \geq \rho_0$*, so this extension also has finite propagation speed. For* $m = 0$*, the correction at its smooth points is*

$$\theta(t-\rho_0) \frac{2\sin(\pi\nu)}{\pi\alpha} e^{\mathrm{i}(n+\beta)(\varphi-\varphi')} \frac{\cosh(\nu u_0)}{4\pi r r' \sinh u_0}.$$

*Proof.* The homogeneous endpoint choice $b = 0$ is diagonalised by the Bessel transform of order $-\nu > -1$, with regularised eigenfunctions $J_{-\nu}(kr)$. The corresponding half-line self-adjoint transform is part of the homogeneous inverse-square analysis [9]. Weber's heat integral replaces $I_\nu$ by $I_{-\nu}$. Now use

$$I_{-\nu}(\zeta) - I_\nu(\zeta) = \frac{2}{\pi}\sin(\pi\nu) K_\nu(\zeta), \qquad K_\nu(\zeta) = \int_0^\infty e^{-\zeta\cosh u} \cosh(\nu u)\, \mathrm{d}u$$

[12, 13]. The angular normalization gives the coefficient in . The heat integral converges superexponentially in $u$, and Laplace inversion as in Theorem gives the wave identity. After compact time testing the $u$-range is bounded, so the correction is globally well defined. Its support is $t \geq \rho_0 \geq d(x, x')$. Positivity follows from Theorem .

## Three distinct holonomy ledgers

### Geometric holonomy

The geometric deck or gluing rotation is $J_\mu$, with the orientation convention in Section . It rotates the developed transverse plane and fixes $t, z$. The spinless, zero-translation comparison in VIII-ST [27, lem:spinless, sec:matching] is consistent with this fixed-worldsheet geometry. It is not a choice of an independent scalar flux.

### Independent Aharonov–Bohm holonomy

A charged complex scalar coupled to an independently specified flat connection has the holonomy $e^{2\pi\mathrm{i}\beta}$ in . The

number $\beta$ is an input. Neither the scalar nature of the field nor the geometric deficit supplies it. The resulting Green operators are exactly those already constructed.

**Integer-spin components induced by the geometry**

For an integer spin-weight component, the flat Levi-Civita connection can induce a nontrivial equivariance in a developing parallel frame. To fix the convention without ambiguity, consider covariant transverse tensors and define their $+s$ component by contraction with $(e_X + \mathrm{i}e_Y)^{\otimes s}$. Since

$$e_X + \mathrm{i}e_Y = e^{\mathrm{i}\alpha\varphi}(e_r + \mathrm{i}e_{\hat{\varphi}}),$$

a single-valued polar component gives

$$A_{+s}(\varphi + 2\pi) = e^{2\pi\mathrm{i}s\alpha}A_{+s}(\varphi) = e^{-\mathrm{i}s\mu}A_{+s}(\varphi).$$

Thus

$$\beta_{+s} \equiv -\frac{s\mu}{2\pi} \bmod 1, \qquad \beta_{-s} \equiv +\frac{s\mu}{2\pi} \bmod 1.$$

Opposite component conventions interchange the signs. For the rough tensor equation

$$-\nabla^a\nabla_a A + m^2 A = F,$$

each parallel spin-weight component obeys the scalar equation. General tensor bundles may contain several spin weights; $\pm s$ denote the specified pure helicity sectors, not an undeclared decomposition of every tensor into only two components.

**Proposition (Geometrically twisted transverse vectors).** *With the componentwise twisted Friedrichs extension, Theorems , , and apply to each sector . For transverse vectors, $\beta_{+1} = \alpha$. In that sector:*

$$\nu_0 = 1, \qquad \nu_{-1} = \frac{1-\alpha}{\alpha} = \frac{\mu}{2\pi\alpha}.$$

*The latter mode is deficient exactly when $\mu < \pi$. The opposite transverse helicity has the same criterion.*

*Proof.* The operator is the scalar flat-connection operator in each parallel component. The deficiency statements are Lemma with $\beta = \alpha$ and with the opposite character.

**Remark (The full vector problem is larger).** The *t*- and *z*-components of a rough vector field are untwisted scalars. Each has a logarithmic deficient mode for every $0 < \mu < 2\pi$. Accordingly, the criterion $\mu < \pi$ concerns the *transverse helicity sectors*, not essential self-adjointness of the full unconstrained vector problem.

No Maxwell constraint, Proca constraint, compatible apex gauge domain, or spinor Green operator is supplied. Spinors additionally require a spin structure. The spinless matching statement of VIII-ST does not choose one. XI-B's flat de Donder gauge fixing is not a substitute for these missing conical domain conditions [27, 28].

### Slicing holonomy is a separate object

VII-F-ST's thm:holonomy concerns an affine word on a particle-supported cellulation [26]. Its event and combinatorial data are not used in any kernel or deficiency calculation here. In particular:

geometric cone holonomy ≠ independent scalar AB input ≠ slicing certificate.

These are scope distinctions, not premises of a proof. No slicing cell count or finite event is imported into the continuum Green operator.

## Supported replacement of flat-completion operators

### The geometric arrival window

**Definition (Angularly small chart).** An angularly small developing chart is an open $U \subset M$ contained in

$$\{r > r_U,\ \varphi \in I\}, \qquad r_U > 0, \qquad |I| < \pi,$$

with a fixed angular lift on $I$, and equipped with the developing map of Section . For its spatial points,

$$|\alpha(\varphi - \varphi')| < \pi\alpha, \qquad |\mathrm{dev}_{\mathrm{sp}} x - \mathrm{dev}_{\mathrm{sp}} x'| = R_0 = d(x, x').$$

The bundle is trivialised by a local parallel frame.

For spatial pairs in such a chart, define

$$T_{\mathrm{arr}}(x,x') = \min\big(\{\rho_0\} \cup \{R_k : k \neq 0, ?|\psi - 2\pi\alpha k| \le \pi\}\big).$$

Including the equality case does not change the threshold: such a term has $R_k = \rho_0$. This definition records all possible additional geometric arrivals, whether or not their total coefficient later cancels.

**Lemma (Uniform lower bound).** *In an angularly small chart,*

$$T_{\mathrm{arr}}(x,x') \ge L_U := 2r_U \sin\left(\frac{\pi\alpha}{2}\right) > 0.$$

*Proof.* For $k \neq 0$ in ,

$$\pi\alpha \le |\psi - 2\pi\alpha k| \le \pi.$$

Consequently,

$$R_k^2 = (r - r')^2 + 4rr' \sin^2\left(\frac{\psi - 2\pi\alpha k}{2}\right) + \Delta z^2 \ge 4r_U^2 \sin^2(\pi\alpha/2).$$

Also $\rho_0 \ge r + r' > 2r_U$.

**Theorem (Scalar kernel replacement).** *Let $B = A_{F,\beta} + m^2$, let $U$ be angularly small, let $f \in C_c^\infty(U; L_\beta)$, and let $\mathcal{O} \subset U$ be open. Transport $f$ in the local parallel frame to the completion and extend it by zero. For $\sigma \in \{+1, -1\}$, assume*

$$\sigma(t - t') < T_{\mathrm{arr}}(x,x') \quad \text{for every} (t,x) \in \mathcal{O}, ?(t',x') \in \operatorname{supp} f.$$

*Then*

$$(G_\sigma^B f)|_{\mathcal{O}} = \left(G_\sigma^{\mathrm{flat}} \operatorname{dev}_* f\right) \circ \operatorname{dev}|_{\mathcal{O}}.$$

*For $E_B$, it suffices to assume*

$$|t - t'| < T_{\mathrm{arr}}(x,x').$$

*A sufficient common condition for both signs and for $E_B$ is that the time extent of $\mathcal{O} \cup \operatorname{supp} f$ be less than $L_U$. The*

*same replacement holds for the* $\mathrm{E}_0$ *extension of Proposition .*

*Proof.* In the parallel chart frame the direct $k = 0$ term is the ordinary flat kernel with unit weight. Every other retarded geometric term is supported at or after $T_{\mathrm{arr}}$, and the retarded diffracted term is supported at or after $\rho_0$. The global distribution prescription in Theorem preserves these support statements at image boundaries. Hence the difference kernel vanishes on the open pair region in for $\sigma = +1$. Time reflection proves the advanced statement with $\sigma = -1$. Integration against $f$ gives .

The absolute-time condition handles both signs. Lemma gives the uniform sufficient condition. The $\mathrm{E}_0$ correction begins at $\rho_0$, so it also vanishes in the same window.

**Remark (A sufficient window, not a universal converse).** The arrival bound is deliberately conservative. Compatible orbifold characters, angular cancellations, or a particular source can eliminate a prospective additional signal. Therefore one cannot assert that every source differs from its flat response immediately outside the window.

If an additional contribution is a nonzero distribution, some compact tests detect it. An uncancelled massless image delta, or a nonzero diffracted contribution, thus prevents a global operator identity with the completion. This is different from a pointwise statement for every source.

**The independent retained-shadow condition**

Theorem is a kernel comparison. A companion calculation also has its own support contract. Write the developed slab as

$$\widehat{M}_{T_{\mathrm{lab}}} = [0, T_{\mathrm{lab}}] \times \mathbb{R}^3.$$

For a retained observation region $\mathscr{O}$, the X-B-ST contract uses

$$D_* = J_\eta^-(\overline{\mathscr{O}}) \cap \widehat{M}_{T_{\mathrm{lab}}} \operatorname{Subset} \operatorname{dev}(U).$$

Tests belong to $\mathscr{O}$; the required coordinate, interaction, and propagation calculations belong to the retained shadow,

or to a specified compact neighbourhood of it [17, def:supported, ass:host].

The IX-A-ST presentation instead makes the reserve explicit:

$$V = (0, T_{\mathrm{lab}}) \times B_{r_0}, \qquad K_* = [0, T_{\mathrm{lab}}] \times \overline{B}_{r_0 + T_{\mathrm{lab}}},$$

inside a developed region containing

$$(-\delta, H) \times B_R, \qquad R > r_0 + T_{\mathrm{lab}}, \quad H > T_{\mathrm{lab}}.$$

Its closure used in the calculation must stay off $\mathscr{W}$ [21, def:support]. Here $r_0, R$ are Euclidean laboratory radii around its developed centre; they are not the polar radius $r$ from the apex.

**Assumption (Conjunctive supported-transfer contract).** A companion calculation is transferred only if:

**1.** all retained data lie in an angularly small chart, with the signature and parallel-frame conventions fixed;

**2.** its full retained shadow, initial-data region, and all intermediate compact source supports lie in a declared compact regular reserve inside that chart;

**3.** each Green operation satisfies the appropriate signed pair window , on a neighbourhood of the compact pair supports needed for its seminorm estimates. For the ball presentation $(*)$, a simple sufficient choice is

$$T_{\mathrm{lab}} < \min\{R - r_0, L_U\},$$

together with the stated time and regularity reserves. If a calculation uses a larger buffered time interval, its actual time extent replaces $T_{\mathrm{lab}}$ in the arrival bound.

The two inequalities in have different functions. The first keeps the required flat causal calculation inside the laboratory. The second excludes intrinsic additional arrivals. Neither implies the other.

**Corollary (Replacement in finite supported expressions).** *Assume Assumption . A finite scalar expression built from compact smooth sources, smooth local differential operators, and the retained retarded or advanced scalar Green operations has the same retained test actions after replacing every flat scalar Green operation by its intrinsic Friedrichs counterpart. The assertion holds in fixed-support smooth seminorms when the window holds*

*on the corresponding neighbourhoods.*

*Smooth supported Cauchy evolution is also replaced: the relevant propagators are* $W_B(t)$ *and* $\dot{W}_B(t)$*, compared with their flat counterparts under the same initial-to-observation window.*

*Proof.* Apply Theorem to the innermost Green operation. Kernel equality on an open pair region persists under finite differentiation. Smooth differential operators with the declared compact coefficient supports produce the same next compact source. Induction proves the statement for the finite expression, with the retained-shadow hypothesis ensuring that no undeclared intermediate value is used.

For fixed-support seminorms, equality on the stipulated neighbourhoods gives equality of all derivatives used in the estimate. For Cauchy evolution, differentiate the positive-time equality for $W_B$, and use its common Cauchy normalization at zero. The data are smooth and compactly supported away from the apex, so these differentiations are legitimate.

This corollary concerns scalar test actions and classical supported evolution. It does not identify global equation quotients, quantum algebras, or products. In particular, Remark is not bypassed.

**Exact companion targets and their stops**

The following ledger records permitted *operator replacement*, not a new proof of each cited companion theorem. For the original real scalar contracts take $\beta = 0$. A twisted charged scalar requires its corresponding complex-bundle formulation.

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X-B-ST, thm:retarded [17] The linear retarded-response test, its support, normalization, and continuity, with the same local coefficient operator and Hessian sign. No transfer of thm:hbar, positive-$\hbar$ claims, a quantum product extension, or a global equation-quotient identification.

X-C-ST, thm:maps, cor:support [18] The stated scalar response coefficients and finite iterations on actual smooth compact intermediate tests, provided every Green operation remains in the retained shadow and window. No second-order claim outside $D_*$, and no construction of a general $T_2$ or retarded-product hierarchy.

X-D-ST, prop:propagation, thm:convergence [19] Fixed-support $O(\varepsilon)$ propagation and sharp-width convergence of representative scalar tests on the retained interval, with uniform support and window reserves. No use of global Huygens propagation or absence of late signals. No sharp-metric admission follows. The corresponding global free-field quotient is not identified by this paper.

V-B, thm:raw [20] The bulk retarded scalar-wave contribution to the raw current, when its component calculation is within the contract. The initial-surface term $\delta(t)j_\Sigma$ requires its own trace and distributional check. Additional image fronts are not removed.

IX-A-ST, def:support, free evolution [21] The retained smooth scalar Cauchy evolution inside $K_*$, with its strict lateral margin. Boundary-sensitive admission is not a conical Hadamard-state construction.

IX-B-ST and IX-C-ST, free scalar parts [22, 23] Only scalar Green operations satisfying the declared supported contract. The triangle calibration, partial products, presentation relations, and quantum closure are not consequences of this replacement.

III-B, thm:Cauchy [25] The supported laboratory scalar solution through the replaced evolution and its declared data. Not a global conical Cauchy-surface theorem and not an enlargement of its relational domain.

endgroup

**IX-D-ST is not an additional transfer target.** Its saturation and support bridge do not themselves construct an intrinsic conical Green operator [24]. This paper's Theorem is a different statement with additional extension and arrival hypotheses. No IX-D-ST product or causal interface is promoted by calling a retained shadow saturated.

**Which flat blocks are being replaced.** XI-B constructs scalar wave inverses on a Minkowski slab and a graviton inverse

$$G_{\mathrm{gr}}^{\pm} = -\mathscr{P} G_0^{\mathrm{flat},\pm},$$

where $\mathscr{P}$ is trace reversal [28, lem:Green, thm:graviton]. X-G-ST's auxiliary–ghost Green matrix likewise belongs to its declared completion [29, ass:host]. The present theorem replaces the scalar equation block only. It does not establish conical graviton, ghost, auxiliary, BRST, or gauge-constraint blocks, nor does it make XI-B's full algebra intrinsic on the cone.

## A finite Enter catalogue of supplied evaluations

This section is analytically independent of the existence proofs. It supplies a finite reporting interface of the I-B/En09-A type [15, 30], not a physical acquisition map for a Green distribution.

### What can be entered in the catalogue

A wave kernel is generally a distribution. Even at a generic spatial pair it contains delta fronts in time. A purported ``kernel value'' on such a front is not a number. We therefore use smeared entries, or values restricted to a declared smooth part of a kernel.

Choose compact smooth sections $a, f$ whose pair supports satisfy the transfer contract, and define

$$Q_{\mathrm{cone}} = \left\langle a, G_+^F f \right\rangle, \qquad Q_{\mathrm{flat}} = \left\langle \mathrm{dev}_* a, G_+^{\mathrm{flat}} \mathrm{dev}_* f \right\rangle.$$

These are finite complex numbers, and Theorem proves their equality. Choose also one fixed generic spatial pair and define

$$C_{\mathrm{dif}}(x, x') = -\frac{\Phi_\beta(0,\theta)}{4\pi^2 \alpha \sqrt{rr'}}, \qquad \kappa_L = \frac{2e^{-\gamma_E}}{L}.$$

A possible finite catalogue is: toprule Entry Mathematical target Required scope metadata

midrule C1 $Q_{\mathrm{cone}}$ $\alpha, \beta, m, B, a, f,$ frame, support window

C2 $Q_{\mathrm{flat}}$ Same tests and normalization as C1

C3 $C_{\mathrm{dif}}(x, x')$ Generic pair, angular lift, front convention

C4 $\kappa_L$, and optionally $m^2 - \kappa_L^2$ Boundary scale $L$, units, mass

bottomrule For a complex entry, its real and imaginary parts are reported separately.

A report word contains a finite rational approximation or enclosure, its error convention, and a status: for example ANALYTIC, SUPPLIED_NUMERIC, or UNCERTIFIED. The existence of an enclosure is not inferred merely by printing a decimal. An installed symbolic formula is also a finite description, not an exact finite-bit encoding of an arbitrary real number.

The number $\kappa_L$ is the magnitude of a transverse negative-energy scale. It is a positivity certificate for the *mass-shifted* operator only when accompanied by the comparison $m \geq \kappa_L$.

**Complete destructive reporting**

**Theorem (Finite supplied-report instrument).** *Let $C \cong \mathbb{C}^D$, $D = 2^q \geq 2$, have a calibrated orthonormal basis $\{|\tau_j\rangle\}$. Let A contain every surviving finite register. Install finite report words $w_j$, with explicit unused and failure labels, and put*

$$D_j = \langle \tau_j |_C \otimes I_A.$$

*Then*

$$\mathscr{M}(X) = \sum_j |j, w_j\rangle\langle j, w_j|_R \otimes D_j X D_j^\dagger$$

*is completely positive and trace preserving and obeys*

$$\mathrm{Tr}_R \mathscr{M}(X) = \mathrm{Tr}_C X.$$

*These identities hold with every untouched finite reference. The quantum output contains A, not C.*

*Proof.* The recorded Kraus maps are $L_j = |j, w_j\rangle_R \otimes D_j$, and

$$\sum_j L_j^\dagger L_j = \sum_j |\tau_j\rangle\langle\tau_j| \otimes I_A = I_{CA}.$$

Expanding

$$X = \sum_{j,k} |\tau_j\rangle\langle\tau_k| \otimes X_{jk}$$

shows that the $j$-th output block is $X_{jj}$. Summing those blocks gives the partial trace. Tensoring the calculation with a reference identity proves the complete statements.

This is the finite instrument of I-B [15, thm:instrument], used in X-B-ST and compared with En09-A [17, 30]. It reports installed data; it does not calculate an unknown kernel by destructive readout. A finite-register input state in this theorem is not a state of the continuum field.

**Proposition (Calibration and recovery limits).** *For the instrument :*

***1.*** *Distinct labels $t_j$ required to equal one fixed reading-event time on every positive-probability branch permit at most one occurring label.*

***2.*** *There is no channel left inverse on the full clock–survivor input space.*

***3.*** *The classical record marginal does not determine the surviving quantum state.*

***4.*** *No finite clock matrices satisfy $[T_C, H_C] = \mathrm{i}\hbar I_C$ with $\hbar \neq 0$.*

*Proof.* Distinctness proves the first assertion. For the second, the two clock states

$$\frac{|\tau_0\rangle + |\tau_1\rangle}{\sqrt{2}}, \qquad \frac{|\tau_0\rangle - |\tau_1\rangle}{\sqrt{2}}$$

with the same surviving preparation have identical destructive outputs. A left inverse cannot recover both. Inputs $|\tau_0\rangle\langle\tau_0| \otimes \rho_A$ and $|\tau_0\rangle\langle\tau_0| \otimes \tilde{\rho}_A$ give the same deterministic record but different residuals when the two states differ. Finally, the trace of a commutator is zero, whereas $\mathrm{Tr}\,(\mathrm{i}\hbar I_C) = \mathrm{i}\hbar D \neq 0$.

These are the fixed-event, no-return, record–residual, and finite-CCR limitations recorded in I-B [15, prop:fixed-event, prop:no-return, prop:record-residual, prop:no-ccr] and in En09-A [30]. They concern the finite interface, not consumption of the geometric time coordinate.

### A fixed-interface table bound

To state a meaningful small approximation bound, the complete output interface must be fixed. Different deterministic decimal strings in orthogonal classical records are perfectly distinguishable; small numerical difference between their printed values does not make those channels close.

For comparison, suppose a bounded real target has been converted to a Bernoulli parameter $p_j \in [0,1]$. Let $\Lambda_p$ first perform the clock dephasing, retain the code and residual, and append a binary report with probability $p_j$ on code $j$.

**Proposition (Complete fixed-table distance).** *For two Bernoulli tables $p, q$ on the same complete interface,*

$$\frac{1}{2}\|\Lambda_p - \Lambda_q\|_\diamond = \max_j |p_j - q_j|.$$

*Thus certified rational approximations of the parameters give exactly the corresponding fixed-table channel bound.*

*Proof.* After adjoining an arbitrary reference and dephasing the clock, the $j$-block of an input operator is $X_{jj}$. The two binary output blocks of the difference are $(p_j - q_j)X_{jj}$ and its negative. Hence their total trace norm is bounded by

$$2\max_j |p_j - q_j| \sum_j \|X_{jj}\|_1 .$$

Pinching the clock is an average of unitary conjugations, so $\sum_j \|X_{jj}\|_1 \le \|X\|_1$. This proves the upper bound, including every reference. A basis-clock input at an index attaining the maximum gives the lower bound.

This is the type of rational-table comparison used in X-A-ST [16, prop:table]. It bounds a finite report channel, not a Green operator in a continuum norm. Any physical transport interpretation additionally needs compatible complete channels, causal locations, durations, storage, and error allowances of the kind stated in Oai283-A [32]. It is not furnished by alone.

### Finite tests do not establish continuum coverage

**Proposition (Coverage and an explicit finite-test limitation).** *Let $\mathcal{V}$ be an independently specified*

*finite-dimensional space containing all residuals admissible for a particular claim. If a finite evaluation map*

$$\mathscr{Q}|_{\mathcal{V}} : \mathcal{V} \to \mathbb{C}^s$$

*is injective, then* $\mathscr{Q}(R) = 0$ *implies* $R = 0$ *within that declared class.*

*Without a coverage hypothesis, finite tests do not in general establish a distributional identity. In particular, any finite family of compact linear evaluations of smooth symmetric tensors can miss a nonzero compactly supported conserved perturbation in a regular developed region.*

*Proof.* The first assertion is injectivity.

For the second, work in a relatively compact regular Minkowski chart and, for $h \in C_c^\infty$ there, put

$$S_{\mu\nu}[h] = (\partial_\mu\partial_\nu - \eta_{\mu\nu}\mathrm{Box}_\eta)h.$$

This tensor is symmetric and conserved. Every one of the finitely many evaluations is a linear functional of $h$. Their common kernel in the infinite-dimensional space of such $h$ contains a nonzero element. Its trace is

$$\eta^{\mu\nu}S_{\mu\nu}[h] = -3\mathrm{Box}_\eta\, h.$$

A compactly supported smooth solution of $\mathrm{Box}_\eta\, h = 0$ has zero Cauchy data before its time support and is zero. Hence the chosen nonzero $h$ has nonzero trace, and $S[h] \neq 0$. Extension by zero preserves smoothness because its support lies strictly inside the regular chart.

The coverage condition and conserved-perturbation example are the finite-test distinction made in Oai282-A [31, prop:coverage, prop:invisible]. The example is not asserted to be a difference of two Green kernels satisfying the same extension condition. It illustrates why a finite comparison, by itself, does not prove a continuum identity.

## Integrity of numerical comparisons

The existence, support, and kernel identities above are analytical statements. Their proofs do not use the numerical discrepancies reported in the supplied draft.

The supplied draft refers to scripts and logs that are not contained in this single-file source. Accordingly:

**1.** the inherited discrepancies are retained as *reported numerical data*, not as newly verified error bounds;

**2.** no execution, independent rebuild, interval certification, or code availability is asserted;

**3.** a submission using those values as reproducible computational evidence should provide the scripts, versions, truncations, precision settings, raw values, and residuals;

**4.** no reported finite agreement is used as a proof of causal support or distributional equality. This values-plus-residual discipline is consistent with the integrity-aware reporting of XVI-O1-ST [33]. Its unrelated algebraic or quantum statuses are not imported.

A useful record for a new comparison is

(parameters, mode value, closed value, absolute residual, relative residual, precision, cutoffs, tail bounds, status).

When the denominator is zero or unreliable, a relative residual should be marked inapplicable. Appendix preserves the principal supplied numbers, including the high-frequency cancellation failure. Theorem , not that table, proves support.

## Status, exclusions, and remaining problems

### Extension status

begingroup 1.12 toprule Restricted extension Nonnegativity of $B$ Kernel representation Spatial propagation

midrule endfirsthead toprule Restricted extension Nonnegativity of $B$ Kernel representation Spatial propagation

midrule endhead bottomrule endfoot

Friedrichs, any $\beta$ Always Theorem Theorem

$L_L$, $v = 0$ Exactly when $m \geq \kappa_L$ Temporal inverses exist; closed conical wave formula not derived here Open here

$E_0$, $0 < v < 1$ Always Proposition Finite speed from its correction support

$E_\lambda$, $0 < \lambda < \infty$ Always Temporal inverses exist; closed conical wave formula not derived here Open here

$E_\lambda$, $\lambda < 0$ Exactly when the mass dominates every transverse $\kappa_j$ Temporal inverses exist; closed conical wave formula not derived here Open here

Non-diagonal transverse $U(2)$, when $d_\beta = 2$ Not analysed here Outside the rotation-diagonal restriction Not analysed here

$k_z$-dependent, non-product, or longitudinally nonlocal conditions Not analysed here Outside Assumption Not analysed here

endgroup

**Remaining analytical questions**

The following remain unresolved in this paper.

**1.** Whether

$$P_m C_c^\infty(M) = \ker E_B \cap C_c^\infty(M)$$

for the extensions under consideration.

**2.** Closed conical wave formulas and spatial finite propagation for the logarithmic family and the general finite nonzero $\lambda$ family.

**3.** The non-diagonal transverse $U(2)$ family and the wider translation-invariant non-product ledger.

**4.** Pointwise image-boundary prescriptions beyond the global distribution identity proved here.

**5.** Spin structures and spinor domains; compatible Maxwell and Proca constraints at the apex.

**6.** Uniform $\mu \to 0$ limits through moving image boundaries and wave fronts.

**7.** Conical states, Hadamard covariances, quantisation, and interacting or time-ordered products. The ordinary pointwise flat limit away from the moving boundaries is discussed in Appendix ; it does not settle O6.

**Standing non-identifications**

• The cone and its extension condition remain prescribed inputs. Symmetry, conservation, and zero trace do not impose Einstein's equation. For example, in a regular flat chart $t_{\mu\nu} = k_\mu k_\nu$, with $k \neq 0$ constant and null, is conserved, symmetric, and traceless, but the local Einstein tensor vanishes while $t_{\mu\nu} \neq 0$. This is the separation recorded in I-B and X-A-ST [15, prop:not-source] [16, prop:noeinstein].

• A Green inverse is not a time-ordered product. The convolution prescription of X-A-conv-ST is not repaired or unfrozen by this paper [34].

• XI-E's named flat-background Hadamard algebra is not a state or Hadamard construction on this cone [35]. O7 remains unresolved here.

• The instantaneous spatial kernel named in Em02-E is not identified with this retarded wave kernel, and no cosmological lapse is constructed [36]. This is an exclusion of an identification, not an imported theorem about Em02-E.

• I-B's finite transport theorem is a finite register operation on a declared route, not the continuum operator $G_\pm$ [15, thm:transport]. Likewise, a complete finite-interface simulation bound is not a continuum kernel estimate.

• No conical graviton, ghost, auxiliary, Maxwell, Proca, or spinor system follows from replacing a scalar wave block.

• The companion charts' Ward, covariance, admission, residual, and product ledgers are unchanged. A documentary separation, source relationship, or shared vocabulary is not a proof-transfer hypothesis.

• XVI-N is not discharged, XVI-O is not updated, and XV-B is not opened by this paper [37, 38, 39]. The scoped O.1 status remains where it was not closed; is not promoted to true. No inherited characteristic or radiative quantum coefficient is admitted.

• No OpenAI-family identifier, finite evaluation family, or transport catalogue is identified with a cone, holonomy, Green distribution, or quantum field.

**Conclusion.** The intrinsic scalar Green operator is constructed after an extension has been supplied. Friedrichs gives intrinsic causal support and an explicit diffraction formula. Agreement with a flat-completion laboratory requires both a retained-support certificate and an arrival window. Finite Enter reporting can preserve supplied values and residuals but neither supplies those analytical hypotheses nor closes the remaining extension, state, gauge, or product problems.

appendix

## Pointwise flat limits and normalization checks

### A restricted flat limit

For the untwisted Friedrichs kernel, fix regular spatial points and a lifted angle that stays a positive distance from $\pm\pi$ as $\alpha$uparrow 1. For $\alpha$ sufficiently close to one, only $k = 0$ belongs to $\mathscr{K}$. Equation gives

$$D_1(u, \theta) = 0.$$

On angular sets bounded away from the image boundaries, differentiation of the closed denominators gives, for $\alpha$ near one, a bound of the form

$$|\Phi_0(u, \theta)| \leq C(1 - \alpha)(1 + u)e^{-u/2}.$$

Consequently dominated convergence in gives the flat heat and positive resolvent kernels at the fixed pair.

For the retarded kernel tested against a fixed compact time test, only a bounded $u$-interval contributes. The same argument applied to the joint definition in Theorem gives convergence as a time distribution at the fixed generic pair; on smooth kernel regions it also gives the ordinary pointwise limit. This is not uniform through a moving image boundary or a moving front.

### Sign and measure checks

The scalar spectral equation is

$$(\partial_t^2 + B)W_B(t) = 0, \qquad W_B(0) = 0, \qquad \dot{W}_B(0) = \mathrm{id}.$$

Therefore

$$(\partial_t^2 + B)[\theta(t)W_B(t)] = \delta(t)\mathrm{id},$$

and

$$(\partial_t^2 + B)[-\theta(-t)W_B(t)] = \delta(t)\mathrm{id}.$$

Using $+\int_t^\infty W_B(t-s)f(s)\,\mathrm{d}s$ would instead give the negative of the required advanced inverse.

The kernels are normalized relative to $\mathrm{d}t\,\mathrm{d}V = c\,\mathrm{d}T\,\mathrm{d}V$. If one instead writes the source integral using $\mathrm{d}T\,\mathrm{d}V$, the corresponding kernel is

$$c\,G_\pm(c(T-T');x,x').$$

No additional factor is needed when the host volume density $c\,\mathrm{d}T\,\mathrm{d}V$ is used.

The transverse spectral statement $\mathrm{spec}_{\mathrm{ess}}(B_\perp) = [0,\infty)$ must not be substituted for the full product statement . A transverse bound state times a plane wave in $z$ is not an $L^2(N)$ eigenfunction.

## Retained numerical data from the supplied draft

**Evidence status.** Every numerical value in this appendix is copied from the supplied draft, not newly computed or verified. The referenced scripts and logs are not included in this source. In particular, the printed discrepancies are not certified upper bounds on total numerical error. The analytical theorems do not depend on them.

### Reported heat and two-dimensional resolvent checks

The supplied draft reports the cases

$$\mu = \pi/2, \quad \mu = 1, \quad \mu = \pi,$$

together with $\beta = 0.3$ at $\mu = \pi/2$, and $\beta = \mu/(2\pi)$ at $\mu = 1$. The last twist is of the geometric-helicity type up to

orientation.

For the angular heat identity , the reported sample set is

$$(\zeta, \psi/(\pi\alpha)) \in \{(0.7, 0.2), (3, 0.9), (8, -0.6)\},$$

giving fifteen parameter points across the five cases. The reported maximum absolute mode-versus-closed discrepancy is

$$1.9 \times 10^{-22}.$$

At $\mu = \pi, \beta = 0$, the reported diffraction contribution is at most $4 \times 10^{-33}$. The exact vanishing in that case is Corollary , not the decimal test.

For the two-dimensional resolvent, the sample parameters are

$$(s, r, r', \Delta\varphi) = (1, 1, 1.5, 0.5), \qquad (2, 0.8, 1, 2.9),$$

with reported maximum absolute discrepancy

$$1.8 \times 10^{-23}$$

over ten points. Raw paired values were not printed in the supplied summary; these maxima are therefore retained as summaries, not reconstructed logs.

The independent mode formula underlying such a check is

$$R_{2D,\beta}(s; x, x') = \frac{1}{2\pi\alpha} \sum_{n \in \mathbb{Z}} e^{\mathrm{i}(n+\beta)(\varphi - \varphi')} I_{\nu_n}(s r_<) K_{\nu_n}(s r_>).$$

For $r \neq r'$ it is absolutely convergent. Other pairs require the usual kernel interpretation or a convergence prescription.

**Reported Laplace-domain wave check**

At $r = 1, r' = 1.5, \Delta\varphi = 2.9, \Delta z = 0$, the supplied draft compares the Laplace transform of the massless retarded

formula with a mode-built longitudinal Fourier integral. It reports:

toprule Case $s$ Reported Laplace value Reported relative discrepancy

midrule $\mu = \pi/2$ 1 $5.54410642265313 \times 10^{-3}$ $1.5 \times 10^{-19}$

$\mu = \pi/2$ 3 $6.28663902035412 \times 10^{-5}$ $4.2 \times 10^{-19}$

$\mu = \pi/2, \beta = 0.3$ 1 $(2.16479458346901 + 1.65283850075681\mathrm{i}) \times 10^{-3}$ $1.6 \times 10^{-19}$

$\mu = \pi/2, \beta = 0.3$ 3 $(3.34886969040763 + 1.77134457766163\mathrm{i}) \times 10^{-5}$ $1.2 \times 10^{-18}$

bottomrule

The stated comparison integral was

$$\frac{1}{\pi}\int_0^{40} R_{2D,\beta}(\sqrt{s^2 + k^2}; x, x')\,\mathrm{d}k.$$

Its finite upper limit introduces a tail error. Without a supplied tail certificate and rounding analysis, the table is a discrepancy comparison, not a rigorous error enclosure. Agreement at these finitely many Laplace parameters is not an application of Laplace-transform uniqueness to all parameters.

**Reported high-frequency consistency check**

At $\mu = \pi/2, \beta = 0$, with $(r, r', \Delta\varphi) = (1, 1.5, 2.9)$, the supplied distance is

$$d = R_0 = 2.22583034991387.$$

For a unique shortest geometric term, the closed resolvent formula and the $K_0$ asymptotic give

$$e^{sd}\sqrt{s}\, R_{2D,0}(s) \to \frac{1}{2\pi}\sqrt{\frac{\pi}{2d}}.$$

This exact leading constant uses the direct $K_0$ contribution and the strictly longer additional arrivals. It does *not* follow from causal support alone.

The supplied draft gives the limiting value $0.133700810575215$ and the following eighty-digit comparison:

toprule $s$ $e^{sd}\sqrt{s}\,R_{2D}$, modes $e^{s(d+0.05)}\sqrt{s}\,R_{2D}$ Reported mode–closed relative discrepancy

midrule 5 0.175771933447982 0.225695630087555 $2.3 \times 10^{-64}$

10 0.154928248627405 0.255433498944321 $3.2 \times 10^{-74}$

20 0.137892303526610 0.374830142960743 $9.2 \times 10^{-67}$

40 0.133685066614491 0.987806456803755 $7.7 \times 10^{-52}$

80 0.133607527312649 7.29472382177342 $3.8 \times 10^{-7}$

bottomrule

The high-$s$ loss is retained, not suppressed. The supplied account attributes it to cancellation in the mode sum and states that an earlier twenty-five-digit run lost all digits at $s = 40$. These are reported diagnostics. Neither the apparently successful rows nor the failed row prove or refute Theorem .

**Reported flat-limit and front samples**

At $\zeta = 3$ and $\psi/(\pi\alpha) = 0.5$, the supplied untwisted flat-limit samples are:

toprule $\mu$ Reported |diffraction|/$S$ Reported $|S - \alpha e^{\zeta\cos\psi}|/S$ Reported $\max_{u\in[0,6]}|\Phi_0|$

midrule 0.1 $4.23448 \times 10^{-4}$ $4.23448 \times 10^{-4}$ $5.08524 \times 10^{-2}$

0.01 $4.42263 \times 10^{-5}$ $4.42263 \times 10^{-5}$ $5.00801 \times 10^{-3}$

0.001 $4.44216 \times 10^{-6}$ $4.44216 \times 10^{-6}$ $5.00080 \times 10^{-4}$

bottomrule

These are compatible with the restricted limit in Appendix ; they do not establish uniform coverage near a boundary.

For the massless diffracted front at

$$\mu = \pi/2, \qquad (r, r', \Delta\varphi, \Delta z) = (1, 1.5, 1, 0.3),$$

the supplied values of $\sqrt{t^2 - \rho_0^2}\, G_+^{\mathrm{dif}}$, at $t - \rho_0 = 10^{-2}, 10^{-4}, 10^{-6}$, are respectively

$$0.0216738, \qquad 0.0229430, \qquad 0.0229563.$$

The supplied comparison coefficient is $0.0229564$. The exact coefficient is .

**No finite-coverage inference.** The finite samples test the stated implementations at the stated inputs. They do not test every angular boundary, every test function, every extension, or every residual distribution. Proposition remains the relevant logical distinction.

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**[20]** N. Cheung, *V: Admissible currents, retarded response, and finite Enter readout (local adaptation on a prescribed conical spacetime)*, source record V-B-v1.tex, Enter Calculus Companion. Used: the bulk retarded-response contract and its separate initial-surface term. https://enter.grok.me/browse/V-B. https://enter.grok.me/browse/V. Companion record; not a Scholar-indexed journal article.

**[21]** N. Cheung, *IX-A-ST: Boundary-sensitive admission of sharp Cauchy insertions on a prescribed conical Enter spacetime*, version 1, Enter Calculus Companion. Used: def:support, the saturated laboratory, and the free-evolution contract. https://enter.grok.me/browse/IX-A-ST. Companion record; not a Scholar-indexed journal article.

**[22]** N. Cheung, *IX-B-ST: Source-locked triangle calibration on a prescribed conical Enter spacetime*, version 1, Enter Calculus Companion. Used: the supported scalar and presentation-transfer boundaries. https://enter.grok.me/browse/IX-B-ST. Companion record; not a Scholar-indexed journal article.

**[23]** N. Cheung, *IX-C-ST: A supported partial product for one sharp ordered word*, version 1, Enter Calculus Companion. Used: the retained scalar comparison prescription and its distinction from a partial product. https://enter.grok.me/browse/IX-C-ST. Companion record; not a Scholar-indexed journal article.

**[24]** N. Cheung, *IX-D-ST*, version 1, Enter Calculus Companion. Used only for the distinction between supported saturation and a global intrinsic Green-operator theorem. https://enter.grok.me/browse/IX-D-ST. Companion record; not a Scholar-indexed journal article.

**[25]** N. Cheung, *III-B: Classical relational descent in a conical Enter laboratory*, source record III-B-v1.tex, Enter Calculus Companion. Used: thm:Cauchy, in its supported laboratory scope. https://enter.grok.me/browse/III-B. Companion record; not a Scholar-indexed journal article.

**[26]** N. Cheung, *VII-F-ST: A particle-supported slicing certificate with a finite spacetime Enter interface*, version 1, Enter Calculus Companion. Used only to distinguish its thm:holonomy from cone holonomy. https://enter.grok.me/browse/VII-F-ST. Companion record; not a Scholar-indexed journal article.

**[27]** N. Cheung, *VIII-ST: Affine matching, persistent cellulation, and certified flow on a prescribed conical Enter spacetime*, version 1, Enter Calculus Companion. Used: lem:spinless, sec:matching; no spin structure or gauge apex condition is imported. https://enter.grok.me/browse/VIII-ST. Companion record; not a Scholar-indexed journal article.

**[28]** N. Cheung, *XI-B*, version 1, Enter Calculus Companion. Used: lem:Green, thm:graviton; the flat multiplet is not transferred wholesale. https://enter.grok.me/browse/XI-B. Companion record; not a Scholar-indexed journal article.

**[29]** N. Cheung, *X-G-ST*, version 1, Enter Calculus Companion. Used: ass:host; its global matrix Green operators belong to the declared flat completion. https://enter.grok.me/browse/X-G-ST. Companion record; not a Scholar-indexed journal article.

**[30]** N. Cheung, *En09-A*, version 1, Enter Calculus Companion. Used: the finite interface discipline and

prop:no-return. https://enter.grok.me/browse/En09-A. Companion record; not a Scholar-indexed journal article.

**[31]** N. Cheung, *Oai282-A*, version 1, Enter Calculus Companion. Used: prop:coverage, prop:invisible; neither is a Green-operator construction. https://enter.grok.me/browse/Oai282-A. Companion record; not a Scholar-indexed journal article.

**[32]** N. Cheung, *Oai283-A*, version 1, Enter Calculus Companion. Used: the complete finite transport contract and resource-accounted comparison, not a continuum kernel estimate. https://enter.grok.me/browse/Oai283-A. Companion record; not a Scholar-indexed journal article.

**[33]** N. Cheung, *XVI-O1-ST*, source record XVI-O1-ST-v5.tex, version 5, Enter Calculus Companion. Used: integrity-aware supplied-report and residual-retention discipline only. https://enter.grok.me/browse/XVI-O1-ST. Companion record; not a Scholar-indexed journal article.

**[34]** N. Cheung, *X-A-conv-ST*, version 1, Enter Calculus Companion. Cited for its retained convolution boundary, not as a premise selecting a Green extension. https://enter.grok.me/browse/X-A-conv-ST. Companion record; not a Scholar-indexed journal article.

**[35]** N. Cheung, *XI-E*, version 1, Enter Calculus Companion. Used only to distinguish its named flat-background Hadamard algebra from a conical state or product construction. https://enter.grok.me/browse/XI-E. Companion record; not a Scholar-indexed journal article.

**[36]** N. Cheung, *Em02-E*, Enter Calculus Companion, series record. Named only as an excluded identification; no theorem from this record is used in a proof. https://enter.grok.me/browse/Em02-E. Companion record; not a Scholar-indexed journal article.

**[37]** N. Cheung, *Landmark XVI-N*, Enter Calculus Companion, series record. Frozen classical insertion; not discharged by this paper. https://enter.grok.me/browse/XVI-N. Companion record; not a Scholar-indexed journal article.

**[38]** N. Cheung, *Landmark XVI-O*, Enter Calculus Companion, series record. Read-only status reference; no O.1

update. https://enter.grok.me/browse/XVI-O. Companion record; not a Scholar-indexed journal article.

**[39]** N. Cheung, *XV-B*, Enter Calculus Companion, series record. Named only as an unchanged downstream boundary. https://enter.grok.me/browse/XV-B. Companion record; not a Scholar-indexed journal article.

**[40]** N. Cheung, *Black-Hole Information as an Access Problem: Enter, Quantum Recovery, and Complete Residuals*, figshare (2026). DOI: 10.6084/m9.figshare.34335024.v1. https://doi.org/10.6084/m9.figshare.34335024.v1.

**[41]** N. Cheung, *Operational Wormhole Certificates—Existence, Dimension-Dependent Scaling, Operational Functions, and Composition*, figshare (2026). DOI: 10.6084/m9.figshare.34549575.v1. https://doi.org/10.6084/m9.figshare.34549575.v1.

**[42]** N. Cheung, *Enter, Quantum Magic, and Quantum Gravity: Readout, Recovery, and the Limits of Geometric Response*, figshare (2026). DOI: 10.6084/m9.figshare.34335582.v1. https://doi.org/10.6084/m9.figshare.34335582.v1.

**[43]** N. Cheung, *Operational Consistency Does Not Select Lorentzian Geometry*, figshare (2026). DOI: 10.6084/m9.figshare.34441758.v1. https://doi.org/10.6084/m9.figshare.34441758.v1.

**[44]** N. Cheung, *Realization of Relational Time by Destructive Enter on a Clock Register*, figshare (2026). DOI: 10.6084/m9.figshare.34442907.v1. https://doi.org/10.6084/m9.figshare.34442907.v1.

**[45]** N. Cheung, ``The Enter Calculus: A Linear Quantum Lambda Calculus with an Operational Characterization of Measurement," *Frontiers in Physics* **14** (2026), article 1954943. DOI: 10.3389/fphy.2026.1954943. https://doi.org/10.3389/fphy.2026.1954943. Interface/operational background only; not a PDE premise.